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SURVEY AND DEVELOPMENT OF FINITE ELEMENTS FOR NONLINEAR STRUCTURAL ANALYSIS

VOLUME II – NONLINEAR SHELL FINITE ELEMENTS

FINAL REPORT

NAS 8-30626

MARCH 16, 1976

Prepared by

BOEING AEROSPACE COMPANY
RESEARCH AND ENGINEERING DIVISION
SEATTLE, WASHINGTON 98124

R. E. Jones, Technical Leader
J. W. Straayer, Project Leader

Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GEORGE C. MARSHALL SPACE FLIGHT CENTER

MARSHALL SPACE FLIGHT CENTER, ALABAMA 35812

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1.0 INTRODUCTION

This document reports on the development of two new shell finite elements for application to large deflection problems. The elements in question are doubly curved and of triangular and quadrilateral planform. They are restricted to small strains of elastic materials, and can accommodate large rotations.

A principal goal of this work has been to obtain accuracy for strongly nonlinear problems commensurate with that which is obtainable in linear finite element work. This particular goal presents a special challenge, because the presence of strong nonlinearities in plate and shell finite elements of conventional formulation cases excessively stiff behavior.

The elements described in this document make use of a new displacement function approach specifically designed for strongly nonlinear problems. In order to accommodate this approach, two further new developments were necessary, relative to strain displacement equations and nonlinear solution procedures. Thus, the developments reported herein represent an attempt at a considerable extension of the current state of the art for nonlinear two dimensional elements.

The two doubly curved shell elements developed, a triangle and a quadrilateral, are based on the relatively simple linear elements of References 2 and 3. The displacement function development for nonlinear applications is based on the beam element formulations of Reference 1. The procedure is called herein the HMN method, in recognition of its first use by the authors of Reference 1. The strain-displacement equations developed are of the shallow shell type specially tailored to the HMN procedure. Additional terms have been included in these equations in an attempt to avoid the large errors characteristic of shallow shell elements in certain types of problems (e.g., Ref. 7). An incremental nonlinear solution procedure specifically adopted to the element formulation was developed. This solution procedure is of combined incremental and total Lagrangian type, and uses a new updating scheme. A computer program was written to evaluate the developed formulations.

This program can accommodate small element groups in arbitrary arrangements. Two simple problems have been successfully solved, and are reported in Section 10. The results indicate that this new type of element has definite promise and should be a fruitful area for further research.

2.0 GEOMETRY AND COORDINATE SYSTEMS

There are three coordinate systems that are required for the two new elements; the local coordinate system, the global coordinate system, and the solution coordinate system. The elemental matrices are derived in the local coordinate system and are eventually merged and solved in the solution coordinate system. With slight variations, these matrices can be discussed for both the triangular and quadrilateral elements. The geometry of the elements is input in the global coordinate system. The triangular element is assumed to be a constant curvature element, whereas the initial geometry of the quadrilateral element has the shape functions of the assumed displacements.

Base planes for the local coordinate systems are established as shown in Figure 1. The triangle has the x axis parallel to the line joining nodes 1 and 2 with positive sense from 1 to 2. The y axis is perpendicular to the x axis and lies in the plane of nodes 1, 2 and 3 with positive sense in the direction of node 3. The z axis is determined by the right hand rule.

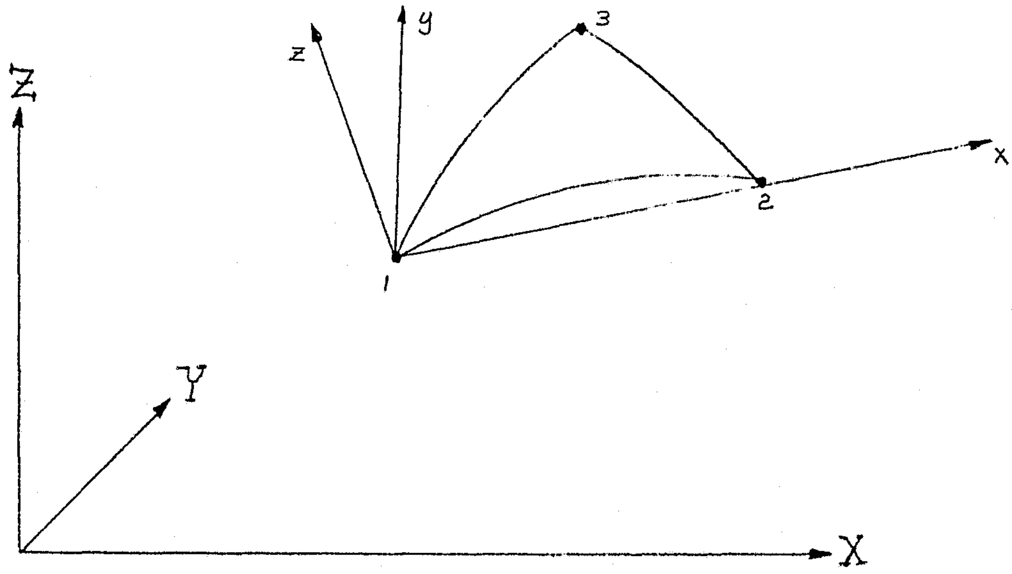
For the quadrilateral, a mean plane is established through the midpoints of the straight lines joining the corner nodes. The x axis is parallel to the line bisecting sides 3,5 and 1,7 with positive sense from side 1,7 to side 3,5. The y axis is perpendicular to the x axis, lies in the mean plane, with positive sense from side 1,3 to side 7,5. The z axis is determined by the right hand rule.

With these definitions, the transformation matrices can be written from the local to the global coordinate system. These transformations are presented in the Appendix, Section 1 2.1. . The transformations that are required, however, are from the local to the solution coordinate system. The solution coordinate system is selected to minimize potential numerical problems by solving the equations in a coordinate system in which the $\hat{z}$ axis is nearly normal to the shell midsurface. This is accomplished by averaging the normal unit vectors of the local coordinate systems of the elements adjoining each

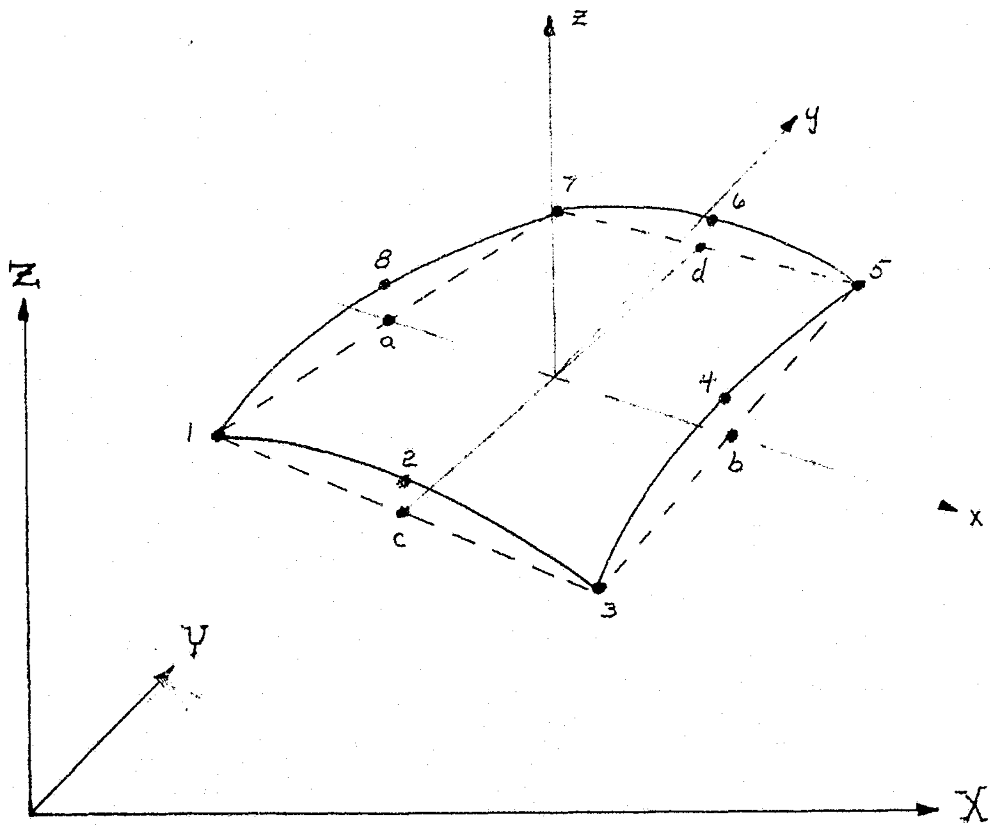
node, Figure 2. The $\hat{x}$ and $\hat{y}$ axes are selected to lie in planes parallel to the XZ and YZ planes, respectively. With these assumptions, the transformation matrix from the local to the solution coordinate system can be written

$$\{q\} = [\Lambda] \{q\} \qquad (2\text{-}1)$$

where the elements of $[\Lambda]$ are given in the Appendix, Section 12.1.



a) TRIANGULAR ELEMENT



b) QUADRILATERAL ELEMENT

Figure 1 : LOCAL AND GLOBAL COORDINATE SYSTEMS

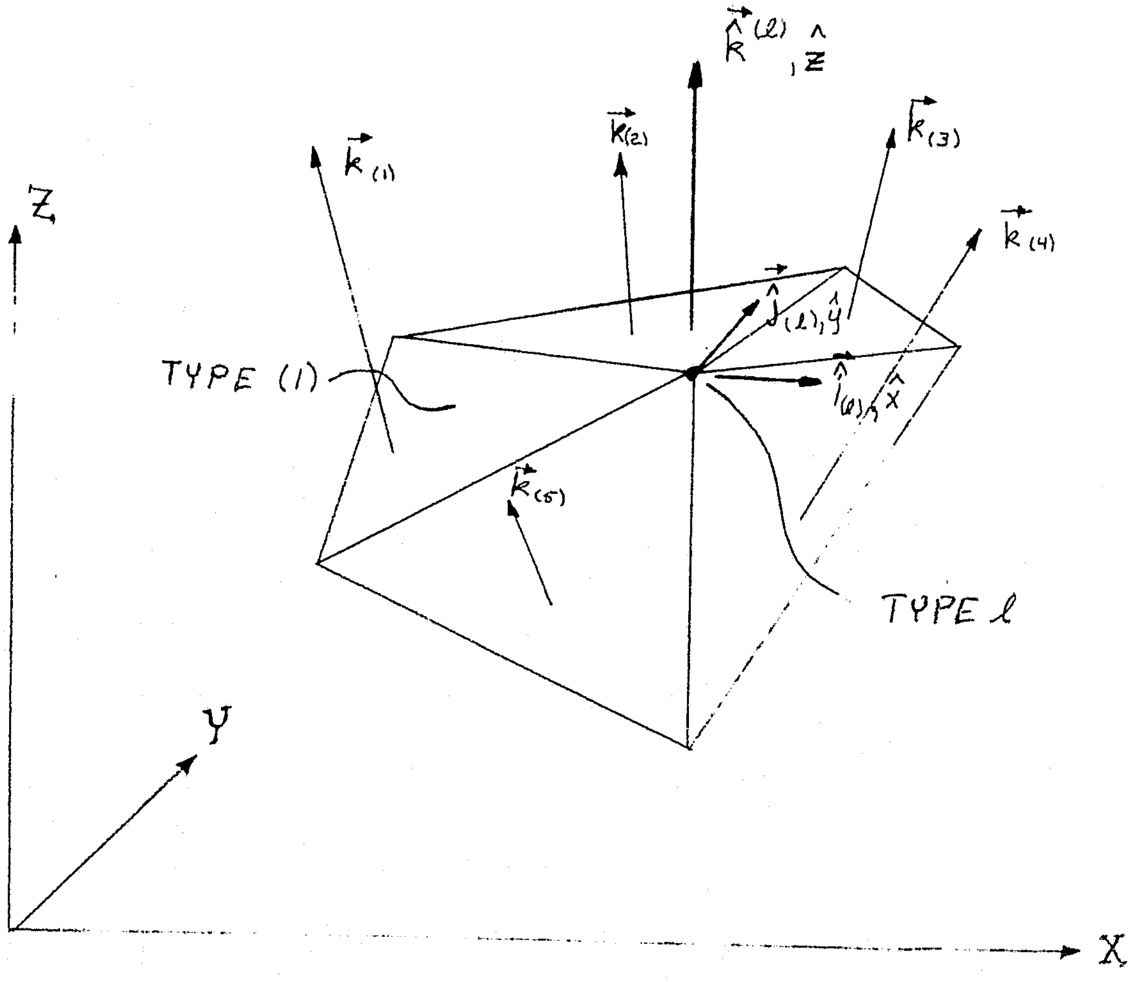


Figure 2: SOLUTION COORDINATE SYSTEM (SHOWN FOR TRIANGULAR ELEMENTS ONLY)

3.0 SHELL STRAIN - DISPLACEMENT RELATIONSHIPS

Shell strain displacement equations take a wide variety of forms, depending on their intended use, coordinate reference systems employed, and specific terms dropped in accordance with thinness or shallowness assumptions. Experience has shown that it is permissible to take considerable liberties in the formulation of these equations, in order to achieve simplicity, so long as the omitted terms are truly negligible for the problem types under consideration. It is important in this regard to perform the mathematical derivation carefully, and to make approximations in the final results, where the full implications of the approximations are visible. This is the approach taken here. For the present work a formulation applicable to stepwise nonlinear analysis of a shell element is required. This application suggests several specific goals for the strain formulation: (1) To handle large rotation effects, the transverse shears should contribute to in-surface, as well as normal-to-surface equilibrium; (2) To avoid undue complexity in the strain equations due to accumulating shell midsurface deformations and associated nonlinearities, element local coordinate systems and incremental displacements should be cartesian; (3) To facilitate implementation of an HMN type constraint procedure, each increment of displacement should involve small rotations, and updating of local cartesian coordinates should be carried out. These goals can be achieved by deriving exact strain displacement equations of a shallow shell element in cartesian coordinates, and dropping appropriate terms in the final result. The approach taken is a combination of the incremental and total Lagrangian approaches since incremental Lagrangian strain increments are used, and in addition total Lagrangian strain is used in equilibrium checks during the nonlinear solution procedure.

Kirchhoff Shell Theory

Consider the shell element shown in Figure 3. The element is referred to two coordinate systems. The x-y-z system is cartesian, and the x-y plane is called the local baseplane of the element. The element in its initial unstrained state is

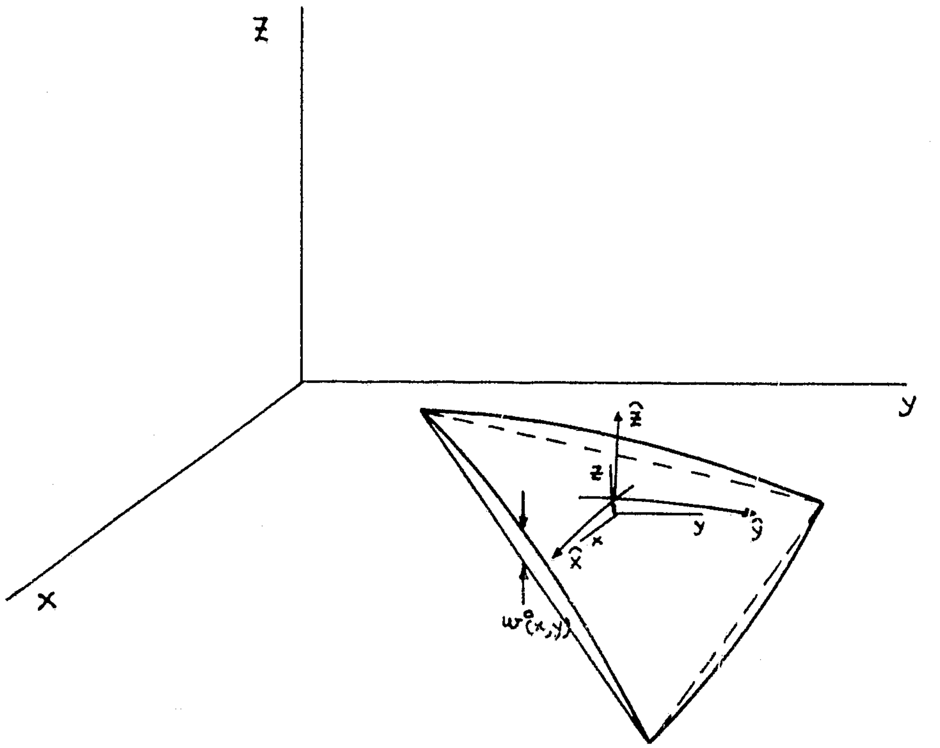


FIGURE 3: SHELL ELEMENT GEOMETRY, BASEPLANE COORDINATE SYSTEM, PROJECTED MIDSURFACE COORDINATE SYSTEM

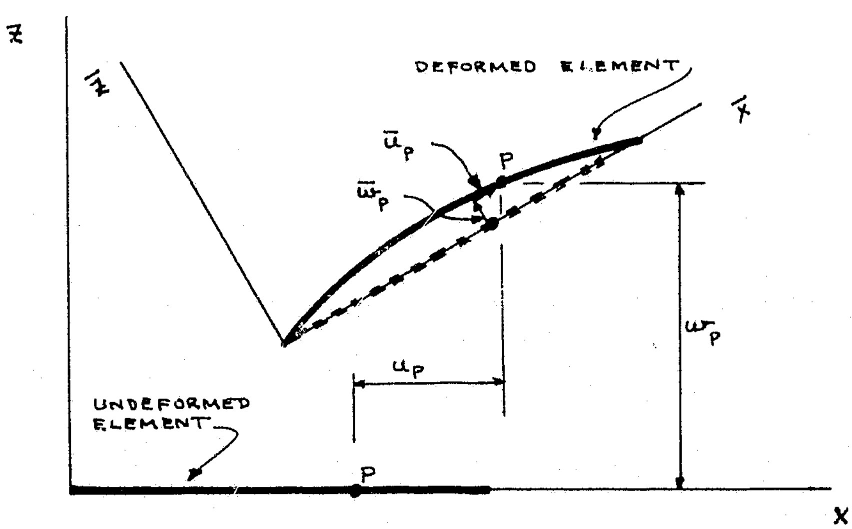


FIGURE 4: LOCAL ELEMENT COORDINATE SYSTEM UPDATING

displaced an amount ω^o from the x-y plane. The projection of the x-y system onto the midsurface of the shell element in its initial position defines the $\hat{x}, \hat{y}, \hat{z}$ system, according to the equations;

$$
\begin{aligned}
x^o &= \hat{x} \\
y^o &= \hat{y} \\
z^o &= \omega^o
\end{aligned}
\qquad (3\text{-}1)
$$

where the superscripts denote the initial state of the element. $\hat{z}$ is perpendicular to $\hat{x}$, $\hat{y}$. It is noted that $\hat{x}$ and $\hat{y}$ are slightly distorted scales of true length. Points of the shell element located a distance $\hat{z}$ above the midsurface of the shell have the x-y-z locations

$$
\begin{aligned}
x^o &= \hat{x} - \hat{z}\,\frac{\partial \omega^o}{\partial \hat{x}} = \hat{x} - \hat{z}\,\omega^{o\prime} \\
y^o &= \hat{y} - \hat{z}\,\frac{\partial \omega^o}{\partial \hat{y}} = \hat{y} - \hat{z}\,\omega^{o\cdot} \\
z^o &= \omega^o + \hat{z}
\end{aligned}
\qquad (3\text{-}2)
$$

After a deflection u, v, w, the coordinates of points on the shell are given by

$$
\begin{aligned}
x &= x^o + u - \hat{z}\,\omega' \\
y &= y^o + u - \hat{z}\,\omega^{\cdot} \\
z &= z^o + \omega
\end{aligned}
\qquad (3\text{-}3)
$$

The directions of u, v, w are, respectively, the positive x, y, z axes. Thus, cartesian displacements rather than the customary ones parallel and normal to the shell surface are employed.

Before deformation, the base vectors of the curvilinear $\hat{x}$, $\hat{y}$, $\hat{z}$ system are

$$
\vec{g}^{\,o}_{\hat{x}} = \frac{\partial \vec{r}^{\,o}}{\partial \hat{x}}, \quad \text{etc.} \qquad (3\text{-}4)
$$

with

$$
\vec{r}^{\,o} = \vec{i}\,x^o + \vec{j}\,y^o + \vec{k}\,z^o = (x^o, y^o, z^o) \qquad (3\text{-}5)
$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit cartesian base vectors,

and, using the above notational shorthand,

$$
\begin{aligned}
\vec{g}^{\,\circ}_{\hat{x}} &= [(1-\hat{z}\,\omega^{\circ''}),\ (-\hat{z}\,\omega^{\circ\cdot'}),\ (\omega^{\circ'})] \\
\vec{g}^{\,\circ}_{\hat{y}} &= [(\hat{z}\,\omega^{\circ'\cdot}),\ (1-\hat{z}\,\omega^{\circ\cdot\cdot}),\ (\omega^{\circ\cdot})] \\
\vec{g}^{\,\circ}_{\hat{z}} &= [(-\omega^{\circ'}),\ (-\omega^{\circ\cdot}),\ (1)]
\end{aligned}
\tag{3-6}
$$

The quantities set off by commas are, respectively, the x, y, and z components of the vectors. After deformation the base vectors are

$$
\begin{aligned}
\vec{g}_{\hat{x}} &= \vec{g}^{\,\circ}_{\hat{x}} + [(u'-\hat{z}\,\omega''),\ (v'-\hat{z}\,\omega^{\cdot'}),\ \omega'] = \vec{g}^{\,\circ}_{\hat{x}} + \vec{h}_{\hat{x}} \\
\vec{g}_{\hat{y}} &= \vec{g}^{\,\circ}_{\hat{y}} + [(u^{\cdot}-\hat{z}\,\omega^{'\cdot}),\ (v^{\cdot}-\hat{z}\,\omega^{\cdot\cdot}),\ \omega^{\cdot}] = \vec{g}^{\,\circ}_{\hat{y}} + \vec{h}_{\hat{y}} \\
\vec{g}_{\hat{z}} &= \vec{g}^{\,\circ}_{\hat{z}} + [(-\omega'),\ (-\omega^{\cdot}),\ (0)] = \vec{g}^{\,\circ}_{\hat{z}} + \vec{h}_{\hat{z}}
\end{aligned}
\tag{3-7}
$$

The metric tensor of the $\hat{x}, \hat{y}, \hat{z}$ coordinate system is defined by the scalar products of the base vectors.

$$g^{\circ}_{ij} = \vec{g}^{\,\circ}_{i} \cdot \vec{g}^{\,\circ}_{j} \tag{3-8}$$

$$g_{ij} = \vec{g}_{i} \cdot \vec{g}_{j} = (\vec{g}^{\,\circ}_{i} + \vec{h}_{i}) \cdot (\vec{g}^{\,\circ}_{j} + \vec{h}_{j}) \tag{3-9}$$

Here, the subscripts denote $\hat{x}$, $\hat{y}$, or $\hat{z}$. It can be shown that for small deformations the strains which occur in passing from the undeformed to the deformed state are described by (Ref. 4)

$$\varepsilon_{ij} = \frac{1}{2}\left(g_{ij} - \overset{\circ}{g}_{ij}\right) \tag{3-10}$$

Thus,

$$\varepsilon_{ij} = \frac{1}{2}\left[\left(\vec{g}_i^{\,\circ} + \vec{h}_i\right)\cdot\left(\vec{g}_j^{\,\circ} + \vec{h}_j\right) - \vec{g}_i^{\,\circ}\cdot\vec{g}_j^{\,\circ}\right] \tag{3-11}$$

which becomes

$$\varepsilon_{ij} = \frac{1}{2}\left[\vec{g}_i^{\,\circ}\cdot\vec{h}_j + \vec{g}_j^{\,\circ}\cdot\vec{h}_i + \vec{h}_i\cdot\vec{h}_j\right] \tag{3-12}$$

Performing the indicated operations yields the results for the strain tensor for a Kirchhoff shell theory

$$\begin{aligned}\varepsilon_{\hat{x}\hat{x}} = u' + \frac{1}{2}\left[(\underline{u'})^2 + (\underline{v'})^2 + (w')^2\right] + w^{\circ\prime} w' \\ - \hat{z}\left[w'' + u'\left(w^{\circ\prime\prime} + w''\right) + v'\left(w^{\circ\cdot\prime} + w^{\cdot\prime}\right)\right. \\ - \hat{z}^2\left[\,.\,.\,.\,.\,.\,.\,\right]\end{aligned} \tag{3-13}$$

$$\begin{aligned}\varepsilon_{\hat{y}\hat{y}} = v^{\cdot} + \frac{1}{2}\left[(\underline{u^{\cdot}})^2 + (\underline{v^{\cdot}})^2 + (w^{\cdot})^2\right] + w^{\circ\cdot} w^{\cdot} \\ - \hat{z}\left[w^{\cdot\cdot} + v^{\cdot}\left(w^{\circ\cdot\cdot} + w^{\cdot\cdot}\right) + u^{\cdot}\left(w^{\circ\cdot\prime} + w^{\cdot\prime}\right)\right] \\ - \hat{z}^2\left[\,.\,.\,.\,.\,.\,.\,\right]\,.\end{aligned} \tag{3-14}$$

$$
\begin{aligned}
2\,\mathcal{E}_{\hat{x}\hat{y}} = 2\mathcal{E}_{\hat{y}\hat{x}} = {} & v' + u^{\cdot} + \underline{u^{\cdot}u'} + \underline{v^{\cdot}v'} + w^{\cdot}w' + \left(w^{\circ\cdot}w' + w^{\circ\prime}w^{\cdot}\right) \\
& - \hat{z}\,[\,2w^{\cdot\prime} + u^{\cdot}\left(w'' + w^{\circ\prime\prime}\right) + u'\left(w^{\cdot\prime} + w^{\circ\cdot\prime}\right) \\
& \quad + v^{\cdot}\left(w^{\cdot\prime} + w^{\circ\cdot\prime}\right) + v'\left(w^{\cdot\cdot} + w^{\circ\cdot\cdot}\right) \\
& - \hat{z}^{2}\,[\,\ldots\ldots\,]
\end{aligned}
\qquad (3\text{-}15)
$$

These equations are to be used incrementally, and the cartesian x, y, z system for each increment will be obtained by updating immediately prior to the increment*. Under these conditions, the displacements u and v will be very small compared to w, and the squares of their derivatives can be neglected. The terms which are neglectable are underlined. However, in the bending terms, the terms involving products of u and v derivatives with w and w^{o} derivatives will be retained, as these terms provide x- and y-direction forces due to transverse shears, the absence of which in shallow shell theory can cause serious errors. With the approximations indicated, the incremental equations are

$$
\begin{aligned}
\Delta\mathcal{E}_{\hat{x}\hat{x}} = {} & \Delta u' + \left(w^{\circ\prime} + w'\right)\Delta w' \\
& - \hat{z}\,\Big[\Delta w'' + \Delta u'\left(w^{\circ\prime\prime} + w''\right) + u'\left(\Delta w''\right) + \Delta v'\left(w^{\circ\cdot\prime} + w^{\cdot\prime}\right) \\
& \quad + v'\left(\Delta w^{\cdot\prime}\right)\Big]
\end{aligned}
\qquad (3\text{-}16)
$$

* This updating is not a change in element shape; details are discussed later in Section 8.

$$\Delta \varepsilon_{\hat{y}\hat{y}} = \Delta v^{\cdot} + (w^{o\cdot} + w^{\cdot})\Delta w^{\cdot} - \hat{z}\left[\Delta w^{\cdot\cdot} + \Delta v^{\cdot}(w^{o\cdot\cdot} + w^{\cdot\cdot}) + v^{\cdot}\Delta w^{\cdot\cdot} + \Delta u^{\cdot}(w^{o\cdot\prime} + w^{\cdot\prime}) + u^{\cdot}\Delta w^{\cdot\prime}\right] \tag{3-17}$$

$$\begin{aligned}\Delta \varepsilon_{\hat{x}\hat{y}} = \Delta \varepsilon_{\hat{y}\hat{x}} = {} & \tfrac{1}{2}(\Delta u^{\cdot} + \Delta v^{\prime}) + \tfrac{1}{2}\Delta w^{\prime}(w^{o\cdot} + w^{\cdot}) + \tfrac{1}{2}\Delta w^{\cdot}(w^{o\prime} + w^{\prime}) \\ & - \hat{z}\Big[\Delta w^{\cdot\prime} + \frac{\Delta u^{\cdot}}{2}(w^{o\prime\prime} + w^{\prime\prime}) + \frac{\Delta u^{\prime}}{2}(w^{o\cdot\prime} + w^{\cdot\prime}) \\ & + \frac{\Delta v^{\cdot}}{2}(w^{o\cdot\prime} + w^{\cdot\prime}) + \frac{\Delta v^{\prime}}{2}(w^{o\cdot\cdot} + w^{\cdot\cdot}) \\ & + \frac{u^{\cdot}\Delta w^{\prime\prime}}{2} + u^{\prime}\frac{\Delta w^{\cdot\prime}}{2} + \frac{v^{\cdot}}{2}\Delta w^{\cdot\prime} + v^{\prime}\frac{\Delta w^{\cdot\cdot}}{2}\Big]\end{aligned} \tag{3-18}$$

In these equations, w denotes the displacement normal to the xy plane, accumulated prior to the incremental displacement under consideration. Note that while the direct strains are actually physical strains, the shear strain tensor values are half of the physical strain.

Consider Figure 4. Here a relatively large displacement of a beam element is shown. The element is approximately inextensional, so that accompanying the large w are necessarily large u displacements. It can be shown easily that $\partial u/\partial x$ is also large. If u and w are referred to the updated base plane which corresponds to the end of the displacement and denoted, respectively by $\bar{u}$ and $\bar{w}$, then $\bar{u}$ becomes small and $\partial\bar{u}/\partial\hat{x}$ becomes negligible. This reasoning justifies the use of the above strain equations, with the underlined terms neglected, for the calculation of total strains corresponding to the end of an increment and the associated updated base plane. This is important to avoid cumulative updating of strains, which can cause relatively large errors.

Shell With Transverse Shear Deformation

Figure 5 illustrates for an x-z section view the deformations considered in this section. Again, u and w are cartesian. The rotation about the y axis, θ_y, controls the x-direction displacements of material points above and below the shell midsurface, according to the right hand rule convention. The coordinates of points on the shell are given by $x^o = \hat{x}$, $y^o = \hat{y}$, $z^o = w^o$, since no θ_x or θ_y need be associated with the initial shape, w^o. After deformation, the material point coordinates are

$$
\begin{aligned}
x &= x^o + u + \hat{z}\,\theta_y \\
y &= y^o + v - \hat{z}\,\theta_x \\
z &= z^o + w + \hat{z}
\end{aligned}
\qquad (3\text{-}19)
$$

The base vectors are

$$
\begin{aligned}
\vec{g}_{\hat{x}} &= \vec{g}^{\,o}_{\hat{x}} + \left[(u' + \hat{z}\,\theta_y'),\ (v' - \hat{z}\,\theta_x'),\ w'\right] \\
\vec{g}_{\hat{y}} &= \vec{g}^{\,o}_{\hat{y}} + \left[(u^{\cdot} + \hat{z}\,\theta_y^{\cdot}),\ (v^{\cdot} - \hat{z}\,\theta_x^{\cdot}),\ w^{\cdot}\right] \\
\vec{g}_{\hat{z}} &= \vec{g}^{\,o}_{\hat{z}} + \left[\theta_y,\ -\theta_x,\ 0\right]
\end{aligned}
\qquad (3\text{-}20)
$$

The strains are

$$
\begin{aligned}
\varepsilon_{\hat{x}\hat{x}} = {} & u' + \tfrac{1}{2}\left[\underline{(u')^2} + \underline{(v')^2} + (w')^2\right] + w' w^{o\prime} \\
& - \hat{z}\left[-\theta_y + u'(w^{o\prime\prime} - \theta_y') + v'(w^{o\cdot\prime} + \theta_x')\right] \\
& - \hat{z}^2\left[\ldots\ldots\right]
\end{aligned}
\qquad (3\text{-}21)
$$

$$
\begin{aligned}
\varepsilon_{\hat{y}\hat{y}} = {} & v^{\cdot} + \tfrac{1}{2}\left[\underline{(u^{\cdot})^2} + \underline{(v^{\cdot})^2} + (w^{\cdot})^2\right] + w^{\cdot} w^{o\cdot} \\
& - \hat{z}\left[\theta_x^{\cdot} + u^{\cdot}(w^{o\cdot\prime} - \theta_y^{\cdot}) + v^{\cdot}(w^{o\cdot\cdot} + \theta_x^{\cdot})\right] \\
& - \hat{z}^2\left[\ldots\ldots\right]
\end{aligned}
\qquad (3\text{-}22)
$$

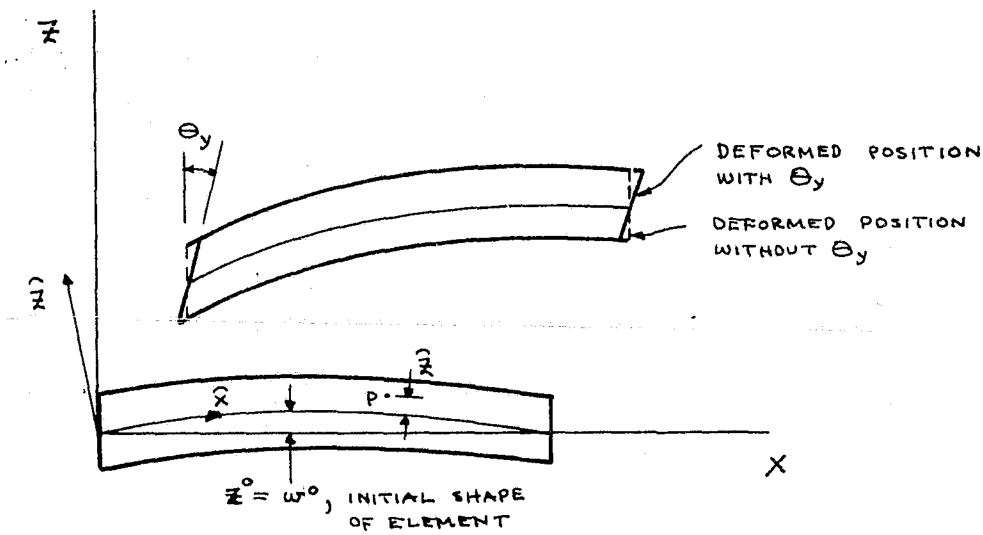


FIGURE 5: LOCAL ELEMENT COORDINATE SYSTEM AND DEFINITIONS FOR ELEMENT WITH SHEAR STRAIN

$$\mathcal{E}_{\hat{x}\hat{y}} = \mathcal{E}_{\hat{y}\hat{x}} = \tfrac{1}{2}\left[u^{\cdot} + v'\right] + \tfrac{1}{2}\left[\underline{u'u^{\cdot}} + \underline{v'v^{\cdot}} + \omega'\omega^{\cdot}\right] + \tfrac{1}{2}\left[\omega^{o\prime}\omega^{\cdot} + \omega^{o\cdot}\omega'\right] \qquad (3\text{-}23)$$

$$- \hat{z}$$

$$-\frac{\hat{z}}{2}\Big[-\theta_y^{\cdot} + \theta_x' + u^{\cdot}\left(\omega^{o\prime\prime} - \theta_y'\right) + u'\left(\omega^{o\cdot\prime} - \theta_y^{\cdot}\right)$$

$$+ v^{\cdot}\left(\omega^{o\cdot\prime} + \theta_x'\right) + v'\left(\omega^{o\cdot\cdot} + \theta_x^{\cdot}\right)\Big]$$

$$\hat{z}^2\left[\ldots\ldots\right]$$

$$\mathcal{E}_{\hat{x}\hat{z}} = \tfrac{1}{2}\left(\theta_y + \omega'\right) + \frac{u'}{2}\left(-\omega^{o\prime} + \theta_y\right) + \frac{v'}{2}\left(-\omega^{o\cdot} - \theta_x\right) - \hat{z}\left[\ldots\ldots\right] \qquad (3\text{-}24)$$

$$\mathcal{E}_{\hat{y}\hat{z}} = \tfrac{1}{2}\left(-\theta_x + \omega^{\cdot}\right) + \frac{u^{\cdot}}{2}\left(-\omega^{o\prime} + \theta_y\right) + \frac{v^{\cdot}}{2}\left(-\omega^{o\cdot} - \theta_x\right) - \hat{z}\left[\ldots\ldots\right] \qquad (3\text{-}25)$$

As before, the underlined terms and the higher order terms in $\hat{z}$ are neglected.

In this non-Kirchhoffian case there exists a transverse shear strain whose contribution to the energy of deformation will have to be handled carefully (Ref. 5), because of its tendency to cause excessive stiffness.

The incremental equations are, dropping the noted terms

$$\Delta\mathcal{E}_{\hat{x}\hat{x}} = \Delta u' + \left(\omega^{o\prime} + \omega'\right)\Delta\omega' - \hat{z}\Big[-\Delta\theta_y' + \Delta u'\left(\omega^{o\prime\prime} - \theta_y'\right) + \Delta v'\left(\omega^{o\cdot\prime} + \theta_x'\right) \qquad (3\text{-}26)$$

$$- u'\Delta\theta_y' + v'\Delta\theta_x'\Big]$$

$$\Delta\mathcal{E}_{\hat{y}\hat{y}} = \Delta v^{\cdot} + \left(\omega^{o\cdot} + \omega^{\cdot}\right)\Delta\omega^{\cdot} - \hat{z}\Big[\Delta\theta_x^{\cdot} + \Delta u^{\cdot}\left(\omega^{o\cdot\prime} - \theta_y^{\cdot}\right) + \Delta v^{\cdot}\left(\omega^{o\cdot\cdot} + \theta_x^{\cdot}\right) \qquad (3\text{-}27)$$

$$- u^{\cdot}\Delta\theta_y^{\cdot} + v^{\cdot}\Delta\theta_x^{\cdot}\Big]$$

$$\Delta\varepsilon_{\hat{x}\hat{y}} = \Delta\varepsilon_{\hat{y}\hat{x}} = \tfrac{1}{2}\left(\Delta u^{\cdot} + \Delta v^{\prime}\right) + \tfrac{1}{2}\left(w^{o\prime} + w^{\prime}\right)\Delta w^{\cdot} + \tfrac{1}{2}\left(w^{o\cdot} + w^{\cdot}\right)\Delta w^{\prime}$$
$$- \hat{z}\Big[\tfrac{1}{2}\left(\Delta\theta_x^{\prime} - \Delta\theta_y^{\cdot}\right) + \tfrac{\Delta u^{\cdot}}{2}\left(w^{o\prime\prime} - \theta_y^{\prime}\right) + \tfrac{\Delta v^{\cdot}}{2}\left(w^{o\cdot\prime} + \theta_x^{\prime}\right) \tag{3-28}$$
$$+ \tfrac{\Delta u^{\prime}}{2}\left(w^{o\cdot\prime} - \theta_y^{\cdot}\right) + \tfrac{\Delta v^{\prime}}{2}\left(w^{o\cdot\cdot} + \theta_x^{\cdot}\right)$$
$$+ \tfrac{1}{2}\left(-u^{\cdot}\Delta\theta_y^{\prime} + v^{\cdot}\Delta\theta_x^{\prime} - u^{\prime}\Delta\theta_y^{\cdot} + v^{\prime}\Delta\theta_x^{\cdot}\right)\Big]$$

$$\Delta\varepsilon_{\hat{z}\hat{x}} = \Delta\varepsilon_{\hat{x}\hat{z}} = \tfrac{1}{2}\left(\Delta\theta_y + \Delta w^{\prime}\right) + \tfrac{\Delta u^{\prime}}{2}\left(-w^{o\prime} - \theta_y\right) + \tfrac{\Delta v^{\prime}}{2}\left(-w^{o\cdot} - \theta_x\right) \tag{3-29}$$
$$+ \tfrac{u^{\prime}}{2}\Delta\theta_y - \tfrac{v^{\prime}}{2}\Delta\theta_x$$

$$\Delta\varepsilon_{\hat{z}\hat{y}} = \Delta\varepsilon_{\hat{y}\hat{z}} = \tfrac{1}{2}\left(-\Delta\theta_x + \Delta w^{\cdot}\right) + \tfrac{\Delta u^{\cdot}}{2}\left(-w^{o\prime} + \theta_y\right) + \tfrac{\Delta v^{\cdot}}{2}\left(-w^{o\cdot} - \theta_x\right) \tag{3-30}$$
$$+ \tfrac{u^{\cdot}}{2}\Delta\theta_y - \tfrac{v^{\cdot}}{2}\Delta\theta_x$$

As discussed above for the Kirchoffian case, total Lagrangian strains can be computed from the above , provided that updating of the element baseplane coordinate system, with the corresponding transformation of the total displacement components, is carried out.

Shallowness and Baseplane Derivatives

Recalling the definition of the coordinate system $\hat{x}$, $\hat{y}$, $\hat{z}$, it is seen that the mapping of the x-y system into the $\hat{x}-\hat{y}-\hat{z}$ system has caused $\hat{x}$ and x, and also $\hat{y}$ and y, to have identical numerical values. Thus, all differentiations with respect to $\hat{x}$ and $\hat{y}$ in the strain equations have identically the same values if they are replaced by, respectively, x and y differentiations. This is required because element displacement functions will be expressed as functions of x and y.

In addition, for the strain tensor values given to be physical strains, it is necessary that the metric tensor components $\overset{o}{g}_{11}$ and $\overset{o}{g}_{22}$ be essentially unity. That is, it is necessary for the lengths in the shell surface to be approximately unity;

$$1 + \left(w^{o\prime}\right)^2 \approx 1 \quad , \quad 1 + \left(w^{o\cdot}\right)^2 \approx 1 \tag{3-31}$$

Thus, in identifying physical values with the strain tensor components, it is necessary that

$$(\omega^{o\prime})^2, (\omega^{o\cdot})^2 <<< 1 \qquad (3\text{-}32)$$

For the work reported herein, this assumption is made based on the fact that each element, referred to its local baseplane, is very shallow, both before and after deformation. Thus, all strain tensor values given are physical values. Note, however, that the physical shear strains are twice the tensor values.

4.0 VIRTUAL WORK FORMULATION

The formulation of the elemental stiffness matrix which includes the HMN degrees of freedom and the degrees of freedom to be reduced by minimum energy considerations, is virtually identical for the triangular and quadrilateral shell elements. The general formulation, proceeding from the virtual work expression, is presented in this section. The details of individual matrices required by the particular element are presented in Sections 6.0 and 7.0.

The virtual work is given by

$$P_i^* \delta q_i^* = \int_A \sigma_i \delta \varepsilon_i \, dA \tag{4-1}$$

where the strain is expressed in terms of the displacement gradients as

$$\varepsilon_i = A0_{ij} \theta_j + \tfrac{1}{2} A1_{ijk} \theta_j \theta_k \tag{4-2}$$

or

$$\delta \varepsilon_i = \left(A0_{ij} + A1_{ijk} \theta_k \right) \delta \theta_j \tag{4-3}$$

For the triangle, the displacements can be written directly in terms of the generalized degrees of freedom. The quadrilateral element, however, requires an additional Jacobian type transformation from the displacement gradients in the parent element to the displacement gradients in the local coordinate system. In general this can be written as

$$\theta_j = \overline{GO}_{jm} q_m^* \begin{cases} \overline{GO}_{jm} = GO_{jm} & \text{for the triangle} \\ \overline{GO}_{jm} = \overline{J}_{j\ell} GO_{\ell m} & \text{for the quadrilateral} \end{cases} \tag{4-4}$$

Thus, the strains can be written

$$\delta \varepsilon_i = \bar{B}_{ir} \delta q_r^* \tag{4-5}$$

$$\bar{B}_{ir} = (AO_{ij} + A1_{ijk}\theta_k)\,GO_{jr}$$
$$= (AO_{ij} + A1_{ijk}\,\overline{GO}_{km}\,q_m^*)\,GO_{jr} \qquad (4\text{-}6)$$

The stresses are

$$\sigma_i = \sigma_i^o + DO_{i\ell}\,\Delta\varepsilon_\ell \qquad (4\text{-}7)$$

where σ^o is the initial stress and $DO_{i\ell}$ is the matrix of material constants. Substituting the above in the virtual work equation yields

$$P_i^* = \int_A (\sigma^o + DO_{s\ell}\,\Delta\varepsilon_\ell)\,\bar{B}_{si}\,dA \qquad (4\text{-}8)$$

The incremental load deflection equation is required

$$\Delta P_i^* = KO_{ij}\,\Delta q_j^* \qquad (4\text{-}9)$$

$$KO_{ij} = \int_A \left\{\bar{B}_{si}\,DO_{s\ell}\,\bar{B}_{\ell j} + \overline{GO}_{mi}\,\sigma_s^o\,A1_{smk}\,\overline{GO}_{kj}\right\} dA \qquad (4\text{-}10)$$

where higher order products of the incremental displacements have been ignored. This gross stiffness matrix includes the deformational degrees of freedom, , plus excess degrees of freedom associated with the minimum energy procedure, α_b , and degrees of freedom associated with the HMN constraints, α_a. Changing from indicial notation to matrix notation, the generalized degrees of freedom can be written in partitioned form as

$$\lfloor q^* \rfloor = \lfloor \delta^* \vdots \alpha_b \vdots \alpha_a \rfloor \qquad (4\text{-}11)$$

The triangular element is derived with several options as to the number of degrees of freedom to be constrained by HMN constraints. Consequently, the generalized displacements α_a assume several forms to account for this variability. This is discussed in greater detail in Section 6.2.

The stiffness equation then has the form

$$\{\Delta p^*\} = [KO]\{\Delta q^*\} \tag{4-12}$$

In general, the HMN constraint can be written as

$$\{\Delta q^*\} = [\tilde{\tilde{C}}]\left\{\frac{\Delta \delta^*}{\Delta \alpha_b}\right\} \tag{4-13}$$

The HMN constrained stiffness matrix is then

$$[\tilde{K}] = [\tilde{\tilde{C}}]^T [K][\tilde{\tilde{C}}] \tag{4-14}$$

Finally, the minimum energy degrees of freedom, α_b, are reduced out by standard procedures to yield the stiffness matrix

$$[K^*] = [\tilde{K}_{\delta^*\delta^*}] - [K_{\delta^* b}][\tilde{K}_{bb}]^{-1}[K_{b\delta^*}] \tag{4-15}$$

This matrix relates the generalized degrees of freedom to the associated generalized forces. For the quadrilateral the generalized degrees of freedom δ^* can be written directly in terms of the nodal degrees of freedom, q. The triangle, however, is derived in terms of the deformational degrees of freedom and these degrees of freedom must be supplemented by the rigid body degrees of freedom

$$\{\delta^*\} = [T^*]\{q\} \tag{4-16}$$

where $[T^*] = [I]$ for the quadrilateral.

Applying this transformation yields the elemental stiffness matrix

$$[K] = [T^*]^T [K^*][T^*] \tag{4-17}$$

relating elemental nodal forces to elemental nodal degrees of freedom.

5.0 HMN Methodology

5.1 Background

The incorporation of nonlinear terms in the strain displacement equations can give rise to problems of excessive stiffness of finite element models for some large deflection problems. This difficulty was discussed first by Mallett (Ref 6), and later by Haftka, Mallett, and Nachbar (Ref 1). These authors found that poor problem solution accuracy was obtained for large deflection and post buckling problems of beams, and formulated a procedure for eliminating the problem. They called the resulting element a "stability element."

The formulation of Haftka, Mallett and Nachbar (HMN method) can be simply explained as follows. The axial strain for small rotations and strains for a beam element is given by the formula.

$$\mathcal{E}_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 \qquad (5\text{-}1)$$

Customarily, u is linear in x, and ω is cubic. The functional form of $\mathcal{E}_x$ is therefore: from u, constant; from ω, through the fourth degree in x. This situation holds also for the linearized incremental case,

$$\Delta \mathcal{E}_x = \frac{\partial \Delta u}{\partial x} + \frac{\partial \omega}{\partial x}\frac{\partial \Delta \omega}{\partial x}, \qquad (5\text{-}2)$$

through the product $\left(\frac{\partial \omega}{\partial x}\right)\left(\frac{\partial \Delta\omega}{\partial x}\right)$.

The result of this situation is that, if the $\mathcal{E}_x$ strains due to $\frac{\partial \omega}{\partial x}$ are retained in their entirety, then superimposed on the constant strain due to $\frac{\partial u}{\partial x}$ are in general, first, second, third, and fourth degree-in-x terms due to $\frac{\partial \omega}{\partial x}$. The strain energy, since it is a

quadratic form, is necessarily increased by these additional terms, even if their average should be zero. This fact, of course, clearly indicates that the element, and associated finite element models, are too stiff.

The formulation given by Haftka, Mallett, and Nachbar avoids this difficulty by introducing supplemental u displacements which include functions of the second through the fifth degree in x. Through the term $\frac{\partial u}{\partial x}$, these functions create strains of the first through fourth degrees, which are sufficient to compensate the element for the higher degree $\left(\frac{\partial w}{\partial x}\right)^2$ type terms. The authors impose the axial equilibrium equation,

$$\frac{\partial \sigma_x}{\partial x} + p_x = 0 \tag{5-3}$$

to solve for the amplitudes of the supplemental u functions. This use of the equilibrium equation is equivalent to minimizing the total potential energy of the element, i.e., to an elemental level reduction of the supplemental u freedoms.

It is noted that in the case of the beam element no inter-element incompatibilities can result from the above equilibrium, or minimum energy, conditions. This results from the fact that the supplemental functions cause no nodal displacement, i.e., the supplemental functions involve only internal displacements. In attempting to apply the HMN procedure to plate or shell elements, this latter convenience does not occur. For these cases, the high degree nonlinear strains due to the w derivatives occur not only within the element, but also on its edges. Thus the supplemental u and v functions must be nonzero both inside the element and on its edges, and the matter of inter-element compatibility must be considered. In addition, the high degree nonlinear

strains which are to be "controlled" by the supplemental functions are now functions of two spatial coordinates instead of only one, with considerable attendant difficulty. For these reasons, the details of the application of the HMN procedure to plate and shell elements are not straightforward. The overall procedure is reasonably clearcut, however, and is briefly explained below.

5.2 Selection of HMN Functions

We began by illustrating the selection of HMN functions by a simple example, and then proceed to a more general discussion. In cartesian coordinates, the nonlinear ω derivative terms are $\left(\frac{\partial \omega}{\partial x}\right)^2$, $\left(\frac{\partial \omega}{\partial y}\right)^2$, and $\left(\frac{\partial \omega}{\partial x}\right)\left(\frac{\partial \omega}{\partial y}\right)$, contributing, respectively, to $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_{xy}$. Similar terms apply to the incremental strains, with the linearized $\Delta\mathcal{E}_x$ term being, for example, $\left(\frac{\partial \omega}{\partial x}\right)\left(\frac{\partial \Delta\omega}{\partial x}\right)$. These types of terms create high degree polynomial strains illustrated by the following several equations, which are representative for a finite element bending formulation

$$
\begin{aligned}
\omega &= (1-y)(1-x^2) \\
\frac{\partial \omega}{\partial x} &= (1-y)(-2x) \\
\left(\frac{\partial \omega}{\partial x}\right)^2 &= (1-2y+y^2)(4x^2) \longrightarrow \mathcal{E}_x \text{ nonlinear}
\end{aligned}
\tag{5-4}
$$

In this example, the nonlinear $\mathcal{E}_x$ is of the second degree in y and x. Presuming that the basic element u displacement function has the form

$$u = (a + bx)(c + dx) \tag{5-5}$$

The basic strain form is

$$\frac{\partial u}{\partial x} = b\,(c + dy) \tag{5-6}$$

Thus the only $\mathcal{E}_x$ strain which can be provided by the basic function is the one linear in y above, and the supplemental u function, here called $\tilde{u}$, must provide the terms listed below if it is to cancel out completely the higher degree nonlinear strains.

$$\tilde{u} \sim x^3,\, yx^3,\, y^2x^3 \tag{5-7}$$

The $\tilde{u}$ functions are of course one degree higher in x than those of $\mathcal{E}_x$.

These $\tilde{u}$ functions are seen to contain the lowest degree in x member, x^3. This is a consequence of the example chosen for illustration, in which the ω function is not complete. In an actual element derivation, ω would contain also linear in x terms, and these in combination with the second degree term would result strain terms containing x and hence the necessity for $\tilde{u}$ terms like x^2.

This simple example suggests a procedure for choosing the polynominal forms of the supplemental $\tilde{u}$ and $\tilde{v}$ functions. The steps involved are:

1. Identify polynomial forms of basic element strain $\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_{xy}$.
2. Identify polynomial forms of nonlinear strains due to $\omega, \Delta\omega$
3. Construct supplemental $\tilde{u}, \tilde{v}$ such that the combinations $u+\tilde{u}, v+\tilde{v}$ contain strains of polynomial degrees through the highest nonlinear forms created by $\omega, \Delta\omega$.

4. Examine the constructed functions to assure that they form a complete set with no intermediate omitted terms, and augment the function set if necessary to assure this.

This last step poses particular difficulties, which are discussed briefly below.

The procedure above can be followed rather easily for any type of element under development. In general, however, it must be followed by an additional task which is highly element dependent, and here lies the essential difficulty in HMN function development. The difficulty lies in the fact that $\tilde{u}$ and $\tilde{v}$ functions must be constructed which, in combination with u and v, form a complete set; and which also do not contribute to the displacements, such as nodal freedoms, which characterize the basic element u and v functions. In general this means that $\tilde{u}$ and $\tilde{v}$ must consist partly of purely internal (pillow) functions, plus other forms which accomplish the needed deformations of the sides of the element, without causing any changes in the values of the basic element freedoms. The pillow functions are needed to make the total $u+\tilde{u}$ and $v+\tilde{v}$ sets complete. The values of $\tilde{u}$ and $\tilde{v}$ along the sides are also subject to the completeness requirement, as well as the requirement to be independent by u and v. Thus the $\tilde{u}$ and $\tilde{v}$ forms, which as developed in the procedure given are simple polynomial forms, must be manipulated into element dependent forms suitable for the particular element under development. This is best illustrated by two examples, given here for functions of a single variable, but capable of generalization to the case of two spatial variables. Figure 6 shows a case in which the basic u function is cubic and $\tilde{u}$ must contain terms through the fifth degree, due to a cubic form of ω. This is a case in which the element freedoms are nodal displacements and first

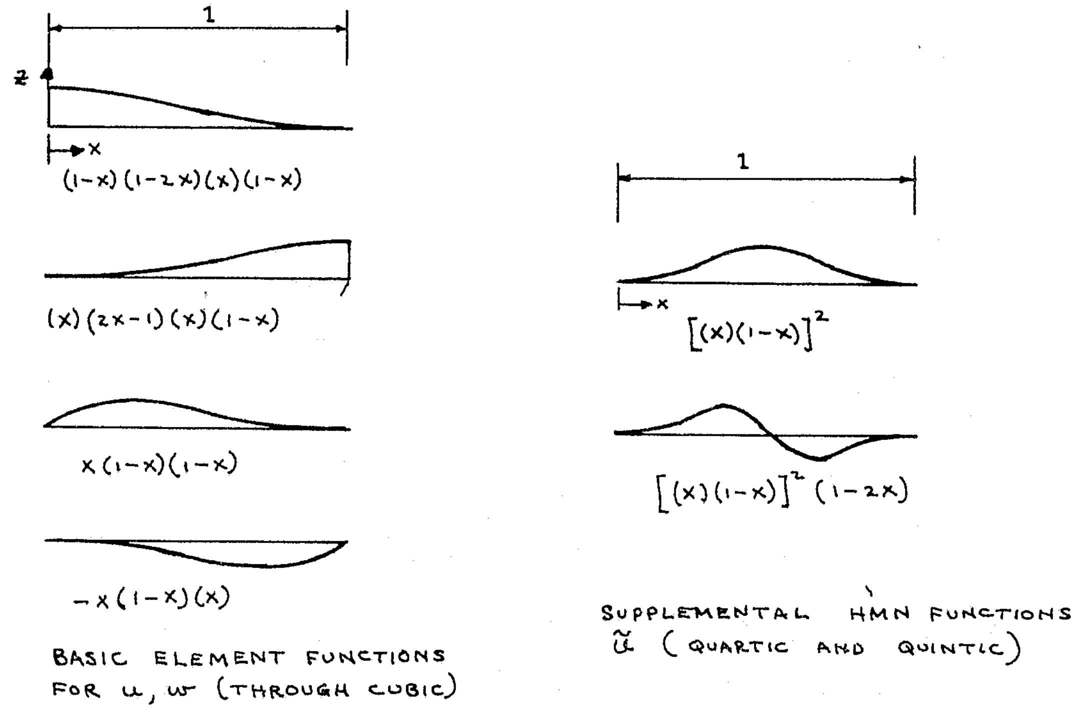


FIGURE 6: ONE-DIMENSIONAL EXAMPLE OF HMN FUNCTIONS FOR ELEMENT WITH DISPLACEMENT AND SLOPE BASIC FREEDOMS

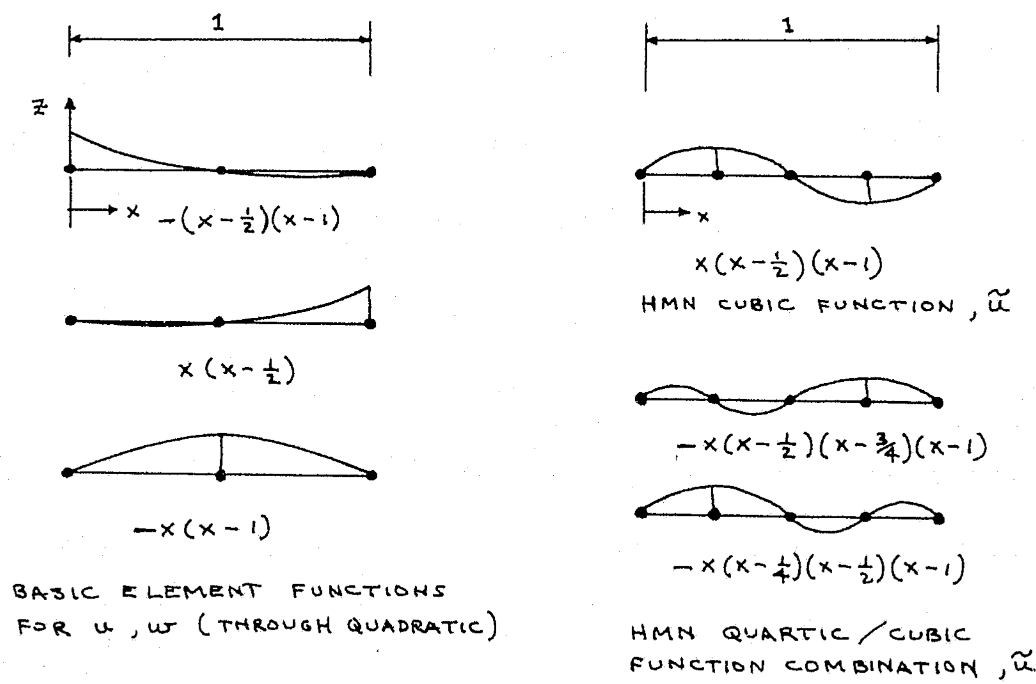


FIGURE 7: ONE-DIMENSIONAL EXAMPLE OF HMN FUNCTIONS FOR ELEMENT WITH DISPLACEMENT FREEDOMS DEFINED AT CORNER PLUS MIDSIDE NODES.

derivatives. The task here is to construct fourth and fifth degree functions which do not affect values of the displacements and their first derivatives at the ends $x = \pm 1$. They have been constructed simply by inspection, and are shown in the figure. Note that all polynominal degrees from the zeroth through the fifth are represented by the combined u and $\tilde{u}$, in order to obtain a complete set of functions.

Figure 7 is a case in which the element basic freedoms make use of a midside node. The basic u and w functions are quadratic, and a cubic $\tilde{u}$ is required. For elements belonging to a family with nodes located along the sides, for example the isoparametric family, it is most convenient to construct the higher degree supplemental functions by conceptually adding more nodes along the sides. Thus, in Figure 7, the basic element has a single midside node and quadratic displacement functions. The addition of the supplemental $\tilde{u}$ is to be accomplished by adding more side nodes. A cubic $\tilde{u}$ is needed, and hence two nodes in the interior of the side would be sufficient. This is not a desirable arrangement, however, because the center node should be retained. The addition of two nodes proves workable. The resulting set of five nodes on a side, including the corner nodes, is capable of representing a fourth degree polynominal, whereas only a cubic is required. This is not a real difficulty, however, and the indicated cubic shape for $\tilde{u}$ provides the necessary function. This shape has the necessary zero displacement of the end and center nodes. Thus, in using additional side nodes as an aid to construct the supplementary functions, it is not necessary to form the highest degree function of which the nodal arrangement is capable.

Suppose as a further example that a quartic were needed. In this case the two quartic diagrams shown in the figure would be used. These two shapes together contain a cubic and a symmetrical quartic, as is easily verified by forming the quantities

$$x(x-\tfrac{1}{2})(x-\tfrac{3}{4})(x-1) \pm x(x-\tfrac{1}{4})(x-\tfrac{1}{2})(x-1) \qquad (5-8)$$

The use of these functions therefore makes unnecessary the explicit use of the cubic form. This example shows the advantage of the procedure of constructing HMN functions by adding midside nodes consistent with the nodal arrangements of an isoparametric family. This procedure has readily provided cubic and quartic forms, while simultaneously omitting all terms of the basic displacement functions, simply by assigning displacements to all nodes other than those which participate in the basic function shapes.

The matter of the required pillow functions is usually resolvable based on consideration of what pillow functions would be required for a higher order finite element whose basic displacement functions include polynominal forms of degree comparable to $\tilde{u}$ and $\tilde{v}$. Thus for the second case above, pillow functions corresponding to a higher order (5 nodes per side) isoparametric element would be used. For the first case above, if for example a triangular element based on area coordinates is being developed, the known pillow functions of higher degree area coordinate forms can be used.

The above discussion has illustrated by example two methods for constructing HMN supplementary functions. One method is applicable to cases in which element freedoms include displacements and displacement derivatives at corner nodes, and the other two cases in which displacements of both corner and intra-side nodes are used as basic

freedoms. In each case, the procedure begins by constructing functions of the needed polynomial degree along the sides of the element, while causing no displacements of the basic element type, and achieves completeness of the $u+\tilde{u}$ and $v+\tilde{v}$ sets by adding pillow functions as needed. This procedure meets the conditions which, at the present stage of development, appear necessary in HMN function methodology.

5.3 HMN Constraints

The added displacement freedoms introduced as HMN functions must be constrained or solved for as a part of the finite element formulation or solution process. There are several alternatives here, and generally the approach to this problem is not straightforward. Some of the relevant details are discussed in this section, and further detail is given in the later sections dealing with the two shell elements developed in the subject research.

The use of the HMN functions is a recognition of the need for highly competent membrane displacement functions in nonlinear bending problems. These functions can be thought of as simply added displacement freedoms, and subjected to inter-element compatibility requirements and solved for in the global solution process. In this case the HMN constraint methodology would not exist as such. However, this approach has the drawback of increasing problem size; it is desirable to keep to a minimum the number of global freedoms, in addition, the philosophy underlying the HMN functions--that these functions are to remove complex forms and keep the strains in the element simple--suggests that the HMN functions be dealt with on the elemental level and that the basic element membrane functions are sufficient in the global solution process.

Elemental level HMN constraints can be of two types: minimum total potential energy (equilibrium); and restrictions on the functional forms of the membrane strains. The two approaches can be used together in a single element derivation, operating on different HMN freedoms. In both cases, consideration must be given to the implied inter-element displacement incompatibilities which are implied by the constraint procedure. The specific strain constraints is considered first.

Suppose that the membrane strains, say ε_x , due to, respectively, the basic element u functions, the HMN $\tilde{u}$ functions, and the bending ω functions are given by

$$\begin{aligned} &\sum_{m=0}^{M_1} \sum_{n=0}^{N_1} u_{mn}\, x^m y^n \quad , \text{and} \\ &\sum_{m=0}^{M_2} \sum_{n=0}^{N_2} \tilde{u}_{mn}\, x^m y^n \quad , \text{and} \\ &\sum_{m=0}^{M_2} \sum_{n=0}^{N_2} \omega_{mn}\, x^m y^n \end{aligned} \tag{5-9}$$

In these forms, x and y are spatial variables and the double subscripted u, $\tilde{u}$, and ω are coefficients which depend on the element freedoms of the type u, $\tilde{u}$, and ω, respectively. The upper limits of the sums have been set by the choices of basic element functions and the HMN functions. Note that for the first summation, the limits M_1 and N_1 are small numbers like 1 or 2, while the values M_2 and N_2 are larger, like 4 or 5, and that these upper limit values are the same for the second and third summations. The latter condition it will be recalled was set by the choice of the HMN functions.

One approach to the HMN constraint procedure is to simply equate coefficients $\tilde{u}_{mn}$ and ω_{mn} for those strain contributions which are of higher degree than M_1 and N_1, i.e.,

$$\begin{aligned} \tilde{u}_{mn} &= -\omega_{mn} \quad , m > M_1 \\ \tilde{u}_{mn} &= -\omega_{mn} \quad , n > N_1 \end{aligned} \tag{5-10}$$

In the first case above, while $m > M_1$, n may be any value; and in the second case the similar condition on m and n occurs. This approach generates a large number of equations and generally involves duplicate constraints, i.e., singularities of the constraint equations. It has been tried and found unworkable in the cases attempted. The procedure amounts to a complete elimination of high degree strains everywhere in the element (above M_1 and N_1 levels). It appears that such an approach would be successful, and it is not clear why it failed to work out. It is recommended that this appraach be given consideration in possible future work on HMN elements, since alternative approaches provide only a partial elimination of the high degree strains, and thus would appear to be inferior.

Another, simpler approach is to eliminate the high degree strains along specific lines in the elements. For example, the high degree x^m forms can be eliminated along a set of lines $y=y_1$, $y=y_2$ etc. This takes the form

$$\sum_{n=0}^{N_2} (\tilde{u}_{mn})x^m + \sum_{n=0}^{N_2} (\omega_{mn} y_1^n) x^m = 0 \; ; \quad m > M_1$$

$$\sum_{n=0}^{N_2} (\tilde{u}_{mn} y_2^n) x^m + \sum_{n=0}^{N_2} (\omega_{mn} y_2^n) x^m = 0 \; ; \quad m > M_1 \tag{5-11}$$

This results in fewer and more easily formulated constraint equations and is a much simpler procedure overall than the one first described. If enough lines, i.e., values y_1, y_2, etc., are used, it should give an excellent control of the high degree strains. In order to relate the two procedures described above, it is noted that if the strain due to $\tilde{u}$ and ω, say ε_x, is known to vary as a quadratic in y, then the constraint should be applied along three y = constant lines, i.e., $y = y_1$, y_2, and y_3 should be used. Fewer y_i values fail to give complete control, and more y_i values necessarily produce singularities in the constraint equations.

This example involves control of the x^m functional forms of the x-direction strain ε_x; the lines in the element are in the same direction as the strain itself. Of course, additionally ε_x could also be controlled in the behavior y^n along specific lines $x = x_1$, x_2, etc. But if the number of y_i lines chosen is adequate to represent the known y functional behavior, as illustrated above for a quadratic form, then this two-direction control of ε_x would likely (or certainly) produce singularities in the constraint equations.

For the control of the shear strain, ε_{xy}, a similar procedure can be used, and in this case the two-directional control has been used without introducing singularities. Thus, for ε_{xy} the polynomial forms y^n are controlled along $x=x_1, x_2 \ldots$, and the forms x^n are controlled along the lines $y=y_1, y_2 \ldots$ This is done for the values of n which exceed those provided by simply the basic element membrane displacement functions.

Control of ε_y follows the analogous pattern to that of ε_x.

The above described procedures refer to rectangular elements, for which the directions of the strains and the directions of the element sides coincide. For triangular elements a procedure has been used in which the HMN functions control strains along the sides of the element and are again handled as described above. Hence, if s denotes the coordinate along a side and n the coordinate normal to a side, the strain ε_s and ε_{sn} can be controlled along all three sides. Isoparametric quadrilateral elements also deal with strain directions which differ from the directions of the element sides. In this case a feasible procedure appears to be to impose the HMN constraints on the "parent" element, and carry the derived constraint equations on for use in the subsequent distorted elements. The above completes the discussion of HMN function constraints based on explicit consideration of the forms of the strain functions. The procedures discussed are illustrative in nature rather than steps of an established methodology. Inventiveness has been found to be required for two dimensional HMN element development.

The imposing of HMN function constraints by means of minimum energy condition is a formal methodology which can be applied without particular difficulty. It is briefly discussed below. This constraint amounts to a

set of equilibrium

conditions, and is therefore the type of approach originally adopted for beams by the authors of References 1 and 6. However, for two dimensional elements it raises questions of inter-element displacement compatibility. These are discussed in Section 5.4.

The minimum energy control of the high degree strains amounts to an elemental level reduction of the $\tilde{u}$ and $\tilde{v}$ freedoms. Thus, if the complete element stiffness matrix is partitioned in the form

$$k = \left[\begin{array}{c|c} k_{aa} & k_{ab} \\ \hline k_{ba} & k_{bb} \end{array}\right] \tag{5-12}$$

where the lower rows pertain to the $\tilde{u}$ and $\tilde{v}$ freedoms, the constrained stiffness matrix refering to the basic elements u, v, and ω functions above has the familiar form

$$\left[k_{aa} - k_{ab}\, k_{bb}^{-1}\, k_{ba}\right] \tag{5-13}$$

This approach tends to produce an overly flexible element due to inter-element displacement incompatibilities.

5.4 Inter-Element Displacement Compatibility

HMN function constraints either by explicit strain function control or by the minimum energy conditions may create inter-element displacement incompatibilities. This difficulty can be largely avoided in the former procedure by a careful selection of the specific strain controls imposed, at the expense of relaxing some of the desired controls

on the high degree strain functions. It cannot be so directly avoided in the minimum energy approach. This section discusses the inter-element compatibility matter and provides recommendations for preferred strain control constraints.

Consider Figure 8 . Line a-b is the side of an element, and coordinate axes x and y are perpendicular and parallel to a-b. The basic element functions are denoted by u, v, and w, and have amplitudes defined by the nodal freedoms. Consider the determination of the supplementary function $\tilde{V}$, which involves displacements parallel to the side a-b. The strain $\mathcal{E}_y$ is completely controlled by the displacements V , $\tilde{V}$ and the slope $\frac{\partial w}{\partial y}$. The value of $\tilde{V}$ determined by HMN constraints imposed on $\mathcal{E}_y$ is completely controlled by $\frac{\partial w}{\partial y}$, since the HMN constraints only apply to those high degree polynomial forms which are unaffected by the lower order basic function V . Hence the $\tilde{V}$ function along a-b will be identical for the elements on the left and on the right of a-b, because $\frac{\partial w}{\partial y}$ is necessarily the same for these two elements. Still further information concerning $\tilde{V}$ is available, however. Since $\tilde{V}$ is determined not only along a-b, but also throughout the adjoining elements, for example by constraints imposed on $\mathcal{E}_y$ along lines a_1-b_1, a_2-b_2 in the figure, it is possible to draw conclusions concerning $\frac{\partial \tilde{V}}{\partial x}$. Not only does the correspondence

$$\left(\tilde{V}\right)_{a-b} \longleftrightarrow \left(\frac{\partial w}{\partial y}\right)_{a-b} \tag{5-14}$$

hold, but, provided $\tilde{V}$ has been given x-direction variation comparable to that of $\frac{\partial w}{\partial y}$, also the first term Taylor expansion holds, and

$$\tilde{V}_{a-b} + \left(\frac{\partial \tilde{V}}{\partial x}\right)_{a-b} \cdot X \longleftrightarrow \left(\frac{\partial w}{\partial y}\right)_{a-b} + \left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\right]_{a-b} \cdot X \tag{5-15}$$

Thus, it can be concluded that

$$\left(\frac{\partial \tilde{v}}{\partial x}\right)_{a-b} \longleftrightarrow \left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\right]_{a-b} \tag{5-16}$$

in which the order of differentiation has been exchanged. Since $\frac{\partial w}{\partial x}$ is continuous across a-b, $\frac{\partial^2 w}{\partial y \partial x}$ is also continuous, it can be concluded that $\frac{\partial \tilde{v}}{\partial x}$ as well as $\tilde{v}$ is continuous across the line a-b. It is noted that $\frac{\partial v}{\partial x}$ can be discontinuous across a-b; this is required to allow equilibrium for the case of a y-direction line load along a-b. The continuity of $\frac{\partial \tilde{v}}{\partial x}$ across a-b will in general be approximate, as it depends on the competence of $\tilde{v}$ as a function of x in each element and also on the exactness with which continuity of $\frac{\partial w}{\partial x}$ is enforced. Nevertheless, clearly $\frac{\partial \tilde{v}}{\partial x}$ tends toward continuity between adjoining elements, across x = constant lines.

Consider next HMN constraints on the shear strain ε_{xy}. The relevant high degree forms contain contributions from $\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$ and from $\frac{\partial \tilde{u}}{\partial y}$ and $\frac{\partial \tilde{v}}{\partial x}$. Continuity between the elements on the left and on the right of line a-b occurs exactly fot $\frac{\partial w}{\partial y}$ and approximately or exactly for $\frac{\partial w}{\partial x}$, depending on the particular finite element formulation employed. Hence $\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}$ is essentially continuous across a-b. It has been seen that $\frac{\partial \tilde{v}}{\partial x}$ is at least approximately continuous, and hence also $\frac{\partial \tilde{u}}{\partial y}$ is continuous across a-b. Since $\frac{\partial \tilde{u}}{\partial y}$ is continuous across a-b, and since $\tilde{u}$ vanishes at the nodes a and b, $\tilde{u}$ itself is necessarily continuous across a-b, or at least approximately so.

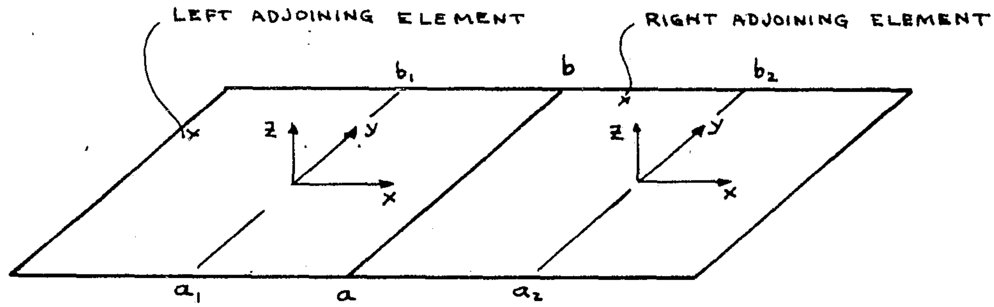


FIGURE 8 : DIAGRAM OF ADJOINING ELEMENTS TO SUPPORT DISCUSSION OF CONTINUITY OF HMN FUNCTIONS $\tilde{u}$ AND $\tilde{v}$ ACROSS INTERELEMENT LINE a-b.

It has been seen that HMN constraints on ε_y and ε_{xy} along x = constant lines provides exact or approximate compatibility of $\tilde{u}$ and $\tilde{v}$ between elements across these same lines as inter-element boundaries. It remains to consider the use of an HMN constraint on ε_x along an x = constant line, for example line a-b in Figure 8 . This strain constraint relates $\frac{\partial \tilde{u}}{\partial x}$ and $\frac{\partial w}{\partial x}$. Since the latter is continuous, or approximately so, across a-b, this constraint makes $\frac{\partial \tilde{u}}{\partial x}$ approximately continuous across a-b. This particular result is not really essential, inter-element continuity of $\tilde{u}$ and $\tilde{v}$ already having been obtained, and hence the use of such an HMN constraint, i.e., on an extensional strain whose fiber direction is normal to the line along which the constraint is imposed, is not recommended. The likelihood of dependent constraint equations together with the lack of basic need for this constraint are the basis for this recommendation.

One further matter concerning explicit strain constraints and inter-element compatibility needs discussion. Consider again the constraint on ε_{xy} along line a-b in Figure 8 . For the higher degree polynomial forms in y, the strain is defined by $\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$ and $\frac{\partial \tilde{u}}{\partial y}+\frac{\partial \tilde{v}}{\partial x}$, and the discussion given earlier applies. However for the highest degree x basic function in ε_{xy} and the lowest degree x HMN function in ε_{xy}, due, respectively, to the terms $\frac{\partial u}{\partial y}$ and $\frac{\partial \tilde{v}}{\partial x}$, there is a special problem. For this particular polynominal degree only, an HMN constraint condition would involve the correspondence

$$\frac{\partial u}{\partial y}+\frac{\partial \tilde{v}}{\partial x} \longleftrightarrow \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \qquad (5\text{-}17)$$

in which both basic functions and HMN functions are involved. The term $\frac{\partial u}{\partial y}$ is not in general continuous across line a-b in the figure, and hence for this particular

polynomial term neither would $\frac{\partial \tilde{v}}{\partial x}$ be continuous. The major difficulty in application of this constraint is that it is inconsistent with the constraint based on $\mathcal{E}_y$ and involving $\frac{\partial \tilde{v}}{\partial y}$ in a form which is a function of x. As discussed earlier, the constraint tends to match up the x-directional functional forms of $\frac{\partial \omega}{\partial y}$ and $\frac{\partial \tilde{v}}{\partial y}$, while imposing equality of their y-direction polynomial coefficients. The lowest degree $\mathcal{E}_{xy}$ HMN constraint terms oppose this x-functional form match-up, and thus can be expected to cause difficulties in the HMN constraint equation set. It is recommended that the lowest degree $\mathcal{E}_{xy}$ HMN constraint be omitted. No inter-element displacement compatibilities will be caused by this omission.

Finally, the matter of constraint of the HMN functions by minimum energy conditions requires discussion. In this procedure, the element is effectively made as flexible as possible by setting the $\tilde{u}$ and $\tilde{v}$ freedoms to values which minimize the strain energy. It is virtually guaranteed that the result of this constraint applied to HMN edge functions will be inter-element displacement discontinuities. The possibility is strong in this case that an excessively flexible element will result. Therefore, minimum energy constraints on HMN functions are only recommended for the internal (pillow function) freedoms of the HMN set.

The above rationale for the inter-element continuity of $\tilde{v}$ due to an $\mathcal{E}_{sn}$ constraint is dependent on the continuity of $\frac{\partial \tilde{u}}{\partial n}$*. If all the HMN constraints are as described, with, in particular, $\mathcal{E}_s$ conditions imposed on lines parallel to the side, this continuity should be obtained. However, it may occur that the internal, pillow-type HMN functions are controlled not by explicit strain constraints but by

* Here $\tilde{v}$ is normal to the side, $\tilde{u}$ is parallel to the side, consistent with definition used in the element derivation.

minimum energy conditions. In this case, if these pillow functions are capable of non-zero $\frac{\partial \tilde{u}}{\partial n}$ on the element edges, then the reasoning leading to continuity of $\tilde{v}$ based on an ε_{sn} constraint becomes less certain. The difficulty is that constraints are operating on $\tilde{u}$ which do not necessarily produce $\frac{\partial \tilde{u}}{\partial n}$ inter-element continuity. It would appear that even in this case, if explicit ε_s constraints are imposed along lines parallel to the sides, then continuity of $\frac{\partial \tilde{u}}{\partial n}$ and hence of $\tilde{v}$, through an ε_{sn} constraint, would be obtained. For the triangular element described in Section 6, however, this is not the case. Here, ε_s constraints are only used along the sides themselves, and minimum energy conditions control the pillow functions. In this case it would have to be determined by numerical experimentation whether the combination of ε_s and ε_{sn} constraints, with minimum energy internal function constraints, yields satisfactory results. In the initial formulation of the triangle done in this contract, not only ε_s and ε_{sn} but also ε_n explicit strain constraints were imposed along the sides. It has been verified that this was unsatisfactory. Inter-element incompatibilities of $\tilde{v}$ occurred, and the resulting element was too flexible. Consequently, $\tilde{v}$ was omitted along the sides of the element.

5.5 Total HMN Functions and Constraints

The foregoing has tacitly dealt with application of the HMN procedures to incremental finite element calculations. A subsequent section of this report will deal with iterative corrections of incremental solutions in order to avoid accumulating errors due to stepwise linearization. It is noted here that the HMN function magnitudes are also subject

to accumulating error, and that they also should be improved through iterative corrections. All the HMN methodology given in the earlier sections applies equally to total or incremental strains, and thus is applicable to iterative calculations.

5.6 Stiffness Matrix Development Sequence

HMN constraints can be applied at the outset of the elemental stiffness matrix calculation, or they can be applied after the elemental matrix is calculated. In the first case, the stiffness matrix never involves, explicitly, the HMN freedoms, as they have been made dependent on the basic element freedoms before the matrix generation. In the second case, the elemental matrix is initially derived in terms of all the u , v , w , $\tilde{u}$ and $\tilde{v}$ freedoms, as though they were independent, and the constraint is accomplished later in a standard stiffness matrix transformation.

Those HMN freedoms which are to be eliminated by minimum energy constraints are always handled in a way similar to the second procedure. The elemental matrix is developed retaining these freedoms, and their elimination is then accomplished by matrix partitioning and a standard elemental level reduction. The second type of procedure has been used in the work of this contract for all HMN function determination.

The HMN equations and their use to constrain the stiffness matrix are outlined briefly below. In the discussion, the unknowns $\{\Delta\alpha_a\}$ are the coefficients of the HMN functions used to obtain explicit control of the strains along the sides of the element. The unknowns $\{\Delta\alpha_b\}$ are the coefficients of HMN functions which are entirely interior to the element, i.e., which vanish on the sides of the element. Since the derivatives of the functions corresponding to the $\{\Delta\alpha_b\}$ do not necessarily vanish on the sides of the element, the specific HMN equations involve $\{\Delta\alpha_b\}$ as well as $\{\Delta\alpha_a\}$. The $\{\Delta q^*\}$ and $\{q^*\}$ contribute the strains due to the bending rotations.

In general, the HMN equations can be put in the form

$$[H]\{\Delta\alpha_a\} = [HO]\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} + [H1(q^*)]\ \{\Delta q^*\} \tag{5-18}$$

where $[H]$ is a square matrix. Consequently,

$$\{\Delta\alpha_a\} = [H]^{-1}[HO]\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} + [H]^{-1}[H1(q^*)]\ \{\Delta q^*\} \tag{5-19}$$

Expand these matrices by adding columns of zeros such that

$$\{\Delta\alpha_a\} = [CO]\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} + [C1(q^*)]\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} \tag{5-20}$$

or

$$\{\Delta\alpha_a\} = [C^*]\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} \tag{5-21}$$

Then expand with the identity matrix to obtain the constraint equation

$$\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b \\ \Delta\alpha_a\end{Bmatrix} = \begin{bmatrix} I \\ \hline C^* \end{bmatrix}\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} = [\approx{C}]\begin{Bmatrix}\Delta q^* \\ \Delta\alpha_b\end{Bmatrix} \tag{5-22}$$

which is used to obtain the HMN constrained stiffness matrix.

$$[\tilde{K}] = [\approx{C}]^T[K][\approx{C}] \tag{5-23}$$

6.0 TRIANGULAR SHELL ELEMENT

The goal in developing the triangular element was to obtain a relatively simple nonlinear shell element incorporating an HMN capability. The HMN methodology requires that relatively high degree membrane displacement functions be used, particularly if the basic element normal displacement functions contain high degree forms. For this reason, a relatively simple basic element was desired, and the BCIZ element was decided upon (Ref 2). Simplifications related to a shallow shell theory but not subject to all of the shallow shell approximations were desired. This was accomplished through the use of a particularly advantageous set of strain displacement equations, described earlier in Section 3.

The HMN methodology was incomplete at the start of this work, having been previously formulated only for beam elements. For this reason, a number of options were retained in the element formulation, relative to HMN functions and constraints. Calculations have been made to evaluate several of these options. The basic approach to the HMN procedure is outlined in Section 5.

6.1 ASSUMED DISPLACEMENT SHAPES

The transverse displacement is assumed to be that of the BCIZ element, Reference 2. The displacement is composed of two parts, a rigid body translation, $\overline{w}$, and a deformational displacement w^*.

$$\omega = \bar{\omega} + \omega^*$$

$$\omega^R = \sum_{i=1}^{3} \omega_i \zeta_i \qquad (6\text{-}1)$$

$$\omega^* = \sum_{i=1}^{6} F_i \theta_{\omega i}^*$$

$$\lfloor \theta_{\omega_1}^* \ \theta_{\omega_3}^* \ \theta_{\omega_5}^* \rfloor = \lfloor \theta_{\omega x_1}^* \ \theta_{\omega x_2}^* \ \theta_{\omega x_3}^* \rfloor$$

$$\lfloor \theta_{\omega_2}^* \ \theta_{\omega_4}^* \ \theta_{\omega_6}^* \rfloor = \lfloor \theta_{\omega y_1}^* \ \theta_{\omega y_2}^* \ \theta_{\omega y_3}^* \rfloor$$

$$F_1 = y_{12}\left(\zeta_1^2 \zeta_2 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right) + y_{13}\left(\zeta_1^2 \zeta_3 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right)$$

$$F_2 = x_{21}\left(\zeta_1^2 \zeta_2 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right) + x_{31}\left(\zeta_1^2 \zeta_3 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right)$$

$$F_3 = y_{23}\left(\zeta_2^2 \zeta_3 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right) + y_{21}\left(\zeta_2^2 \zeta_1 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right)$$

$$F_4 = x_{32}\left(\zeta_2^2 \zeta_3 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right) + x_{12}\left(\zeta_2^2 \zeta_1 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right)$$

$$F_5 = y_{31}\left(\zeta_3^2 \zeta_1 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right) + y_{32}\left(\zeta_3^2 \zeta_2 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right)$$

$$F_6 = x_{13}\left(\zeta_3^2 \zeta_1 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right) + x_{23}\left(\zeta_3^2 \zeta_2 + \tfrac{1}{2} \zeta_1 \zeta_2 \zeta_3\right)$$

The functions F_i have the characteristic that they are zero at each of the nodes and the derivatives with respect to x and y are zero at each of the nodes except at a node where $\theta_{\omega y_i}^*$ or $\theta_{\omega x_i}^*$ is nonzero.

Along each side of the element w is a complete cubic, taking the form of the conventional beam bending displacement shape expressed in terms of area coordinates. The interior cubic forms are an incomplete cubic. This results from a constraint imposed in Reference 2 to assure that constant curvature states are available.

The membrane type displacements u and v have the same form as the transverse displacement w with some additions. The basic constant strain state $\bar{u}, \bar{v}$ is added to a deformational state u^*, v^*. In addition, there are interior fifth order pillow functions u^I, v^I and fourth and fifth order edge functions u_m^e, v_m^e where m denotes an edge. Referring to Figure 9 , the displacements are written

$$
\begin{aligned}
u &= \bar{u} + u^* + u^I + \lfloor J_u \rfloor \begin{Bmatrix} u_m^e \\ v_m^e \end{Bmatrix} \\
v &= \bar{v} + v^* + v^I + \lfloor J_v \rfloor \begin{Bmatrix} u_m^e \\ v_m^e \end{Bmatrix}
\end{aligned}
\qquad (6\text{-}2)
$$

$$
\begin{aligned}
\bar{u} &= \sum_{i=1}^{3} u_i \zeta_i \\
\bar{v} &= \sum_{i=1}^{3} v_i \zeta_i \\
u^* &= \sum_{i=1}^{6} F_i \theta_{ui}^* \\
v^* &= \sum_{i=1}^{6} F_i \theta_{vi}^* \\
u^I &= \sum_{i=1}^{6} a_i G_i \\
v^I &= \sum_{i=1}^{6} b_i G_i \\
&\left.\begin{aligned} u_m^e &= \sum_{k=1}^{2} c_{km} H_{km} \\ v_m^e &= \sum_{k=1}^{2} d_{km} H_{km} \end{aligned}\right\}
\end{aligned}
\qquad (6\text{-}3)
$$

$$\lfloor J_u \rfloor = \lfloor \cos\alpha_1 \quad -\sin\alpha_1 \quad \cos\alpha_2 \quad -\sin\alpha_2 \quad \cos\alpha_3 \quad -\sin\alpha_3 \rfloor$$

$$\lfloor J_v \rfloor = \lfloor \sin\alpha_1 \quad \cos\alpha_1 \quad \sin\alpha_2 \quad \cos\alpha_2 \quad \sin\alpha_3 \quad \cos\alpha_3 \rfloor$$

$$\lfloor u_m^e \; v_m^e \rfloor = \lfloor u_1^e \quad v_1^e \quad u_2^e \quad v_2^e \quad u_3^e \quad v_3^e \rfloor$$

$$\lfloor \theta_{u_1}^* \; \theta_{u_3}^* \; \theta_{u_5}^* \rfloor = \lfloor \theta_{ux_1}^* \quad \theta_{ux_2}^* \quad \theta_{ux_3}^* \rfloor$$

$$\lfloor \theta_{u_2}^* \; \theta_{u_4}^* \; \theta_{u_6}^* \rfloor = \lfloor \theta_{uy_1}^* \quad \theta_{uy_2}^* \quad \theta_{uy_3}^* \rfloor$$

$$\lfloor \theta_{v_1}^* \; \theta_{v_3}^* \; \theta_{v_5}^* \rfloor = \lfloor \theta_{vx_1}^* \quad \theta_{vx_2}^* \quad \theta_{vx_3}^* \rfloor$$

$$\lfloor \theta_{v_2}^{*} \; \theta_{v_4}^{*} \; \theta_{v_6}^{*} \rfloor = \lfloor \theta_{vy_1}^{*} \; \theta_{vy_2}^{*} \; \theta_{vy_3}^{*} \rfloor$$

$a_i \,,\, b_i \,,\, C_{km} \,,\, d_{km}$ are arbitrary constants

$$
\begin{aligned}
G_1 &= \zeta_1^2 \zeta_2^2 \zeta_3 \\
G_2 &= \zeta_1 \zeta_2^2 \zeta_3^2 \\
G_3 &= \zeta_1^2 \zeta_2 \zeta_3^2 \\
G_4 &= \zeta_1^3 \zeta_2 \zeta_3 \\
G_5 &= \zeta_1 \zeta_2^3 \zeta_3 \\
G_6 &= \zeta_1 \zeta_2 \zeta_3^3 \\
H_{11} &= \zeta_2^2 \zeta_3^2 \\
H_{21} &= \zeta_2^2 \zeta_3^2 (\zeta_2 - \zeta_3) \\
H_{12} &= \zeta_3^2 \zeta_1^2 \\
H_{22} &= \zeta_3^2 \zeta_1^2 (\zeta_3 - \zeta_1) \\
H_{13} &= \zeta_1^2 \zeta_2^2 \\
H_{23} &= \zeta_1^2 \zeta_2^2 (\zeta_1 - \zeta_2)
\end{aligned}
\qquad (6-4)
$$

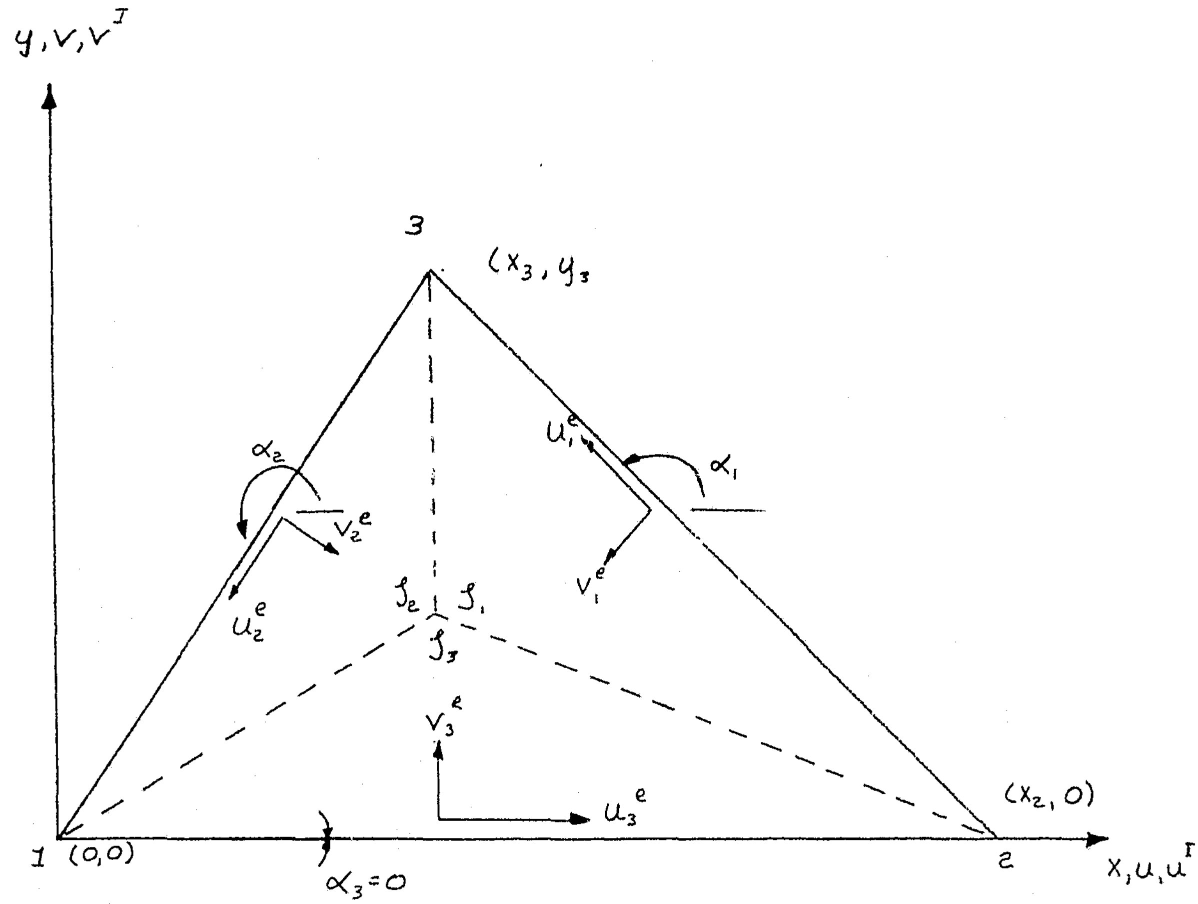


FIGURE 9: DEFINITIONS AND SIGN CONVENTIONS FOR MEMBRANE DISPLACEMENTS FOR TRIANGLE

The HMN functions are required to control fourth degree strains, and hence need to be of the fifth degree in the area coordinates. Along the sides, the needed forms are

$$(\zeta_m \zeta_n)^2 \tag{6-5}$$

and

$$(\zeta_m \zeta_n)^2 (\zeta_m - \zeta_n) \tag{6-6}$$

where m, n take the values 1, 2, 3 and $m \neq n$. These functions have zero values and zero derivatives at the nodes, and thus do not interfere with the basic element freedoms. Each function also is non-zero on only a single side. For example, $(\zeta_1 \zeta_2)^2$ and $(\zeta_1 \zeta_2)^2 \cdot (\zeta_1 - \zeta_2)$ are non-zero on only the side opposite the number 3 node.

The interior functions required to complete the displacement set through the fifth degree are

$$\zeta_m \zeta_n \zeta_p^3 \tag{6-7}$$

and

$$\zeta_m^2 \zeta_n^2 \zeta_p \tag{6-8}$$

in which $m \neq n \neq p$ and the indices take the values 1, 2, 3. These functions and their first derivatives are zero at all the nodes.

The interior functions describe membrane displacements in the element local coordinate directions. However, the edge functions are used to define displacements referred to coordinate axes which are local to the sides themselves. Thus, s and n denote, respectively, local side coordinates along and normal to a side, and $\tilde{u}_p^e$ and $\tilde{V}_p^e$ denote the HMN edge functions parallel and normal to the p th side. The side functions $(\zeta_m \zeta_n)^2$ and $(\zeta_m \zeta_n)^2 (\zeta_m - \zeta_n)$ are used to create four functions; two each for each of $\tilde{u}_p^e$ and $\tilde{V}_p^e$, where the subscript of p denotes the side opposite

node p , and $p \neq m \neq n$. These edge HMN functions have non-zero values along a single side and decrease to zero at the other sides. Each side in turn has such functions, aligned parallel and normal to the side. Clearly, no generality is lost in choosing the side-oriented directions; they are a matter of convenience in implementation of the explicit HMN constraints.

There are a total of 24 HMN functions, of which 12 are interior, and 12 are side functions. Note all of these functions are always used in calculations, however. The computer program has various options on HMN function retention and constraints. In particular, the side functions $\tilde{V}_p^e$ are in some cases omitted.

6.2 ELEMENT DERIVATION

The basic triangular element is derived with several options for handling the excess generalized degrees of freedom. Both minimum energy constraints and HMN constraints are applied to the excess degrees of freedom to reduce the gross 51 by 51 stiffness matrix to the final 27 by 27 elemental stiffness matrix. The program options designate which, if any, of the excess degrees of freedom will be HMN constrained, with the remaining degrees of freedom reduced through minimum energy. There are presently four such options available. This will be clarified as the derivation proceeds.

The virtual work expressions presented in Section 4.0 are applicable to both the triangular and quadrilateral shell elements. The stiffness matrix was obtained in a general form in Equation (4-10), which is repeated here for clarity.

$$K_{ij} = \int_A \left\{ \bar{B}_{si}\, DO_{s\ell}\, \bar{B}_{\ell j} + \overline{GO}_{jr}\, \overset{\circ}{\sigma}_s\, A1_{sjk}\, GO_{kj} \right\} dx\, dy$$
$$\bar{B}_{ir} = \left(AO_{ij} + A1_{ijk}\, \theta_k \right) GO_{jr} \qquad (6-9)$$

Particular application of this equation to the triangular element displacement functions, Section 6.1, and the details of the derivation are contained in this section.

From Equation (4-4), the GO matrix has the form

$$\{\theta\} = [GO]\{q^*\} \qquad (6-10)$$

$$\lfloor \theta \rfloor = \lfloor u_x,\, u_y,\, v_x,\, v_y,\, \omega_x,\, \omega_y,\, \omega_{xx},\, \omega_{yy},\, \omega_{xy} \rfloor$$
$$\lfloor q^* \rfloor = \lfloor \delta^* \vdots \alpha_b \vdots \alpha_a \rfloor$$
$$\lfloor \delta^* \rfloor = \lfloor u_1\ \theta^*_{ux1}\ \theta^*_{uy1}\ v_1\ \theta^*_{vx1}\ \theta^*_{vy1}\ \omega_1\ \theta^*_{\omega x1}\ \theta^*_{\omega y1}$$
$$u_2\ \theta^*_{ux2}\ \theta^*_{uy2} \cdots \qquad \theta^*_{\omega y2}$$
$$u_3\ \theta^*_{ux3}\ \theta^*_{uy3} \cdots \qquad \theta^*_{\omega y3} \rfloor \qquad (6-11)$$
$$\lfloor \alpha_b \rfloor = \lfloor \alpha_1 \rfloor,\ \text{or}\ \lfloor \alpha_1 \vdots \alpha_2 \rfloor,\ \text{or}\ \lfloor \alpha_1 \vdots \alpha_2 \vdots \alpha_3 \rfloor$$
$$\lfloor \alpha_a \rfloor = \lfloor \alpha_2 \vdots \alpha_3 \rfloor,\ \text{or}\ \lfloor \alpha_3 \rfloor\ \text{or}\ \lfloor NULL \rfloor$$

$$\lfloor \alpha_1 \rfloor = \lfloor a_4 \; b_4 \; a_5 \; b_5 \; a_6 \; b_6 \rfloor$$

$$\lfloor \alpha_2 \rfloor = \lfloor d_{11} \; d_{12} \; d_{21} \; d_{22} \; d_{13} \; d_{23} \; a_1 \; b_1 \; a_2 \; b_2 \; a_3 \; b_3 \rfloor \qquad (6\text{-}12)$$

$$\lfloor \alpha_3 \rfloor = \lfloor c_{11} \; c_{21} \; c_{12} \; c_{22} \; c_{13} \; c_{23} \rfloor$$

The form of the GO matrix is given in the Appendix, Section 12.2. Similarly, the A0 and A1 matrices for the triangle can be directly written from Equation(4-2) as

$$\Delta E_i = \left(A0_{ij} + A1_{ijk} \theta_k \right) \Delta \theta_j \qquad (6\text{-}13)$$

where

$$\lfloor \Delta E \rfloor = \lfloor \Delta E_{xx} \, \Delta E_{yy} \, \Delta E_{xy} \, \Delta K_{xx} \, \Delta K_{yy} \, \Delta K_{xy} \rfloor \qquad (6\text{-}14)$$

and $\lfloor \theta \rfloor$ was defined in Equation(6-11). Explicit forms for A0 and A1 are given in the Appendix, Section 12.3. The curvatures (K_{xx}, K_{yy}, K_{xy}) are related to the bending strains by $E_{xx} = Z K_{xx}$ *etc.* Also, physical strains are used instead of tensor strains.

Finally, the stress-strain relation is given in Equation(4-7) as

$$\sigma_i = \sigma_i^0 + DO_{i\ell} \, \Delta E_\ell \qquad (6\text{-}15)$$

where

$$\lfloor \sigma \rfloor = \lfloor N_x \; N_y \; N_{xy} \; M_x \; M_y \; M_{xy} \rfloor$$

$$\lfloor \sigma^0 \rfloor = \lfloor N_x^0 \; N_y^0 \; N_{xy}^0 \; M_x^0 \; M_y^0 \; M_{xy} \rfloor \qquad (6\text{-}16)$$

and the matrix of material constants [DO] is given in the Appendix, Section 12.4.

With these matrices, the gross stiffness matrix of Equation (6-9) can be generated. This matrix is of 51 by 51 order involving the deformational degrees of freedom $\lfloor \delta^* \rfloor$ and the generalized degrees of freedom $\lfloor \alpha_b \vdots \alpha_a \rfloor$. At this point the concept of HMN type constraints is introduced. The HMN constraints are applied to the $\lfloor \alpha_a \rfloor$ degrees of freedom and result in equations of the form

$$[H] \{\Delta \alpha_a\} = [HO] \begin{Bmatrix} \Delta \theta_w^* \\ \Delta \alpha_b \end{Bmatrix} + [H1(\theta_w^*)] \{\Delta \theta_w^*\} \tag{6-17}$$

where

$$\lfloor \theta_w^* \rfloor = \lfloor \theta_{wx1}^*, \theta_{wy1}^*, \theta_{wx2}^*, \theta_{wy2}^*, \theta_{wx3}^*, \theta_{wy3}^* \rfloor \tag{6-18}$$

As indicated in Section 5.6, these equations can be put in the form

$$\begin{Bmatrix} \Delta \delta^* \\ \Delta \alpha_b \\ \Delta \alpha_a \end{Bmatrix} = [\tilde{\tilde{C}}] \begin{Bmatrix} \Delta \delta^* \\ \Delta \alpha_b \end{Bmatrix} \tag{6-19}$$

and the HMN constrained stiffness matrix is then Equation (6-20).

$$[\tilde{K}] = [\tilde{\tilde{C}}]^T [K] [\tilde{\tilde{C}}] \tag{6-20}$$

Four options are available for the constraint procedure as indicated in Equations (6-11) and (6-12). All of the $\lfloor \alpha_a \rfloor$ and $\lfloor \alpha_b \rfloor$ degrees of freedom are eliminated either through HMN constraints or minimum energy reduction.

Option 0: $\begin{cases} \underset{1 \times 0}{\lfloor \alpha_a \rfloor} = \lfloor NULL \rfloor \\ \underset{1 \times 24}{\lfloor \alpha_b \rfloor} = \lfloor \alpha_1 \vdots \alpha_2 \vdots \alpha_3 \rfloor \end{cases}$ No HMN constraints (6-21)

Option 1: $\begin{cases} \underset{1 \times 6}{\lfloor \alpha_a \rfloor} = \lfloor \alpha_3 \rfloor \\ \underset{1 \times 18}{\lfloor \alpha_b \rfloor} = \lfloor \alpha_1 \vdots \alpha_2 \rfloor \end{cases}$ Only normal strain constraints enforced (6-22)

Option 2: $\begin{cases} \underset{1 \times 18}{\lfloor \alpha_a \rfloor} = \lfloor \alpha_2 \vdots \alpha_3 \rfloor \\ \underset{1 \times 6}{\lfloor \alpha_b \rfloor} = \lfloor \alpha_1 \rfloor \end{cases}$ Normal, tangent, and shear strain constraints enforced. (6-23)

Option 3: $\begin{cases} \lfloor \alpha_a \rfloor = \lfloor \alpha_3 \rfloor & \text{Only normal strain constraints enforced} \\ \underset{1\times 18}{\lfloor \alpha_b \rfloor} = \lfloor \underset{1\times 6}{\alpha_1} \mid \underset{1\times 6}{0} \mid \underset{1\times 6}{a_1\, b_1\, a_2\, b_2\, a_3\, b_3} \rfloor & \end{cases}$ (6-24)

For Option 3, the six generalized degrees of freedom corresponding to the normal edge displacement, V^e, are set equal to zero, i.e., d_{11}, d_{12}, d_{21}, d_{22}, d_{31}, d_{32} are zero. The remaining set of displacement functions $\bar{u}, u^*, u^I, u^e, \bar{v}$ v^*, v^I when used in conjunction with the ε_s HMN constraint yields a fully compatible element. The gross stiffness matrix is of order 45 by 45 when the null rows and columns have been deleted.

Options 0, 1 and 3 are subsets of Option 2 ; consequently the most general constraints will be developed and the subsets will be obvious.

The constraint equations are obtained by the procedure developed in Section 5.0. As applied to the triangle, the incremental membrane strains along the three edges can be written

$$\Delta E_n^{(M)} = \frac{\partial \Delta V^e_{(M)}}{\partial n^{(M)}} - \sin\alpha_M \frac{\partial \Delta u^I}{\partial n^{(M)}} + \cos\alpha_M \frac{\partial \Delta V^I}{\partial n^{(M)}} + \left[\frac{\partial \omega^\circ}{\partial n^{(M)}}\frac{\partial \Delta\omega}{\partial n^{(M)}} + \frac{\partial \omega}{\partial n^{(M)}}\frac{\partial \Delta\omega}{\partial n^{(M)}}\right] \tag{6-25}$$

$$\Delta E_s^{(M)} = \frac{\partial \Delta u^e_{(M)}}{\partial s^{(M)}} + \frac{\partial \omega^\circ}{\partial s^{(M)}}\frac{\partial \Delta\omega}{\partial s^{(M)}} + \frac{\partial \omega}{\partial s^{(M)}}\frac{\partial \Delta\omega}{\partial s^{(M)}} \tag{6-26}$$

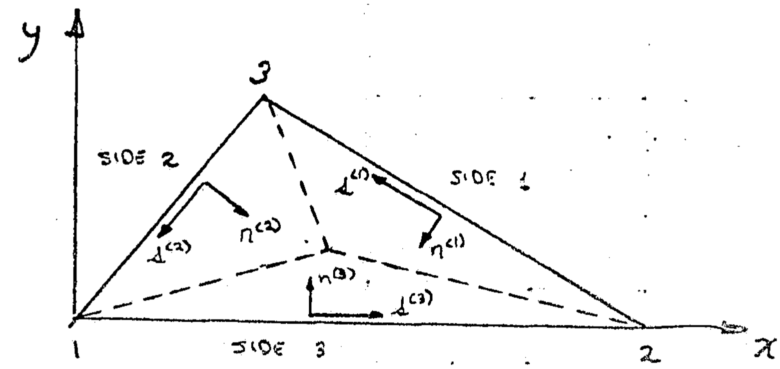
$$\begin{aligned}\Delta E_{sn}^{(M)} = {} & \frac{\partial \Delta u^e_{(M)}}{\partial n^{(M)}} + \frac{\partial \Delta V^e_{(M)}}{\partial s^{(M)}} + \cos\alpha_M \frac{\partial \Delta u^I}{\partial n^{(M)}} - \sin\alpha_M \frac{\partial \Delta u^I}{\partial s^{(M)}} + \sin\alpha_M \frac{\partial \Delta V^I}{\partial n^{(M)}} \\ & + \cos\alpha_M \frac{\partial \Delta V^I}{\partial s^{(M)}} + \frac{\partial \omega^\circ}{\partial s^{(M)}}\frac{\partial \Delta\omega}{\partial n^{(M)}} + \frac{\partial \omega^\circ}{\partial n^{(M)}}\frac{\partial \Delta\omega}{\partial s^{(M)}} + \frac{\partial \omega}{\partial s^{(M)}}\frac{\partial \Delta\omega}{\partial n^{(M)}} \\ & + \frac{\partial \omega}{\partial n^{(M)}}\frac{\partial \Delta\omega}{\partial s^{(M)}}\end{aligned} \tag{6-27}$$

where S and n denote edge tangents and normals, M in parenthesis indicates an edge as illusted in Figure 9 . Application of Equation (6-26) to the three edges of the triangle and equating to zero the multipliers of the terms $\mathfrak{J}^3_{M+1}$ and $\mathfrak{J}^4_{M+1}$ yields two equations per edge or a total of six equations for the $\lfloor \alpha_a \rfloor$ unknowns (along edge M, $\mathfrak{J}_M = 0$ and $\mathfrak{J}_{M+1} + \mathfrak{J}_{M+2} = 1$, therefore the equations can be expressed in terms of the one variable $\mathfrak{J}_{M+1}$). Similarly, application of Equations (6-25) and (6-27) on the three edges and equating to zero the multipliers of the terms $\mathfrak{J}^3_{M+1}$ and $\mathfrak{J}^4_{M+1}$ yields an additional twelve equations. The basis for these equations is summarized in Table 1 . With the derivatives of the area coordinates with respect to the edge coordinates summarized in Table 2 and the displacements defined in Equations (6-1), (6-3), a major algebraic effort yields the HMN equations in the form of Equation (6-17). These equations are presented in the Appendix, Section 12.5. Application of these constraints yields the stiffness matrix in the form of Equation (6-20).

Table 1 : HMN EQUATIONS

Unknowns	Strain	Edge	Equate to Zero the Coefficient of Area Coordinate
α_a	$E_n^{(1)}$	1	ζ_2^3
α_a	$E_n^{(1)}$	1	ζ_2^4
α_a	$E_n^{(2)}$	2	ζ_3^3
α_a	$E_n^{(2)}$	2	ζ_3^4
α_a	$E_n^{(3)}$	3	ζ_1^3
α_a	$E_n^{(3)}$	3	ζ_1^4
α_b	$E_s^{(1)}$	1	ζ_2^3
α_b	$E_s^{(1)}$	1	ζ_2^4
α_b	$E_s^{(2)}$	2	ζ_3^3
α_b	$E_s^{(2)}$	2	ζ_3^4
α_b	$E_s^{(3)}$	3	ζ_1^3
α_b	$E_s^{(3)}$	3	ζ_1^4
α_b	$E_{sn}^{(1)}$	1	ζ_2^3
α_b	$E_{sn}^{(1)}$	1	ζ_2^4
α_b	$E_{sn}^{(2)}$	2	ζ_3^3
α_b	$E_{sn}^{(2)}$	2	ζ_3^4
α_b	$E_{sn}^{(3)}$	3	ζ_1^3
α_b	$E_{sn}^{(3)}$	3	ζ_1^4

TABLE 2: TRANSFORMATION EQUATIONS FOR CALCULATING HMN DERIVATIVES ON ELEMENT SIDES



$n_i^{(3)}$ Means $n^{(3)}$ Evaluated at "node i"

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} X & N_1^{(i)} & S_1^{(i)} \\ X & N_2^{(i)} & S_2^{(i)} \\ X & N_3^{(i)} & S_3^{(i)} \end{bmatrix} \begin{Bmatrix} 1 \\ s^{(i)} \\ n^{(i)} \end{Bmatrix}$$

X - Ignore These Terms for Evaluating Derivatives

	Side 1, i = 1	Side 2, i = 2	Side 3, i = 3
$N_1^{(i)}$	0	$n_2^{(1)}$	$-n_3^{(3)}$
$N_2^{(i)}$	$-n_1^{(1)}$	0	$n_3^{(3)}$
$N_3^{(i)}$	$n_1^{(1)}$	$-n_2^{(2)}$	0
$S_1^{(i)}$	$s_3^{(1)}$	$-s_2^{(2)}$	$s_3^{(3)}-s_2^{(3)}$
$S_2^{(i)}$	$s_1^{(1)}-s_3^{(1)}$	$s_1^{(2)}$	$-s_3^{(3)}$
$S_3^{(i)}$	$-s_1^{(1)}$	$s_2^{(2)}-s_1^{(2)}$	$s_2^{(1)}$

The remaining degrees of freedom other than the BCIZ or HMN constrained degrees are then eliminated through minimum energy reduction.

$$[K^*] = [\tilde{K}_{\delta^*\delta^*}] - [\tilde{K}_{\delta^* b}][\tilde{K}_{bb}]^{-1}[\tilde{K}_{b\delta^*}] \tag{6-28}$$

where the order of $[\tilde{K}_{bb}]$ is to be 24 x 24, 18 x 18, or 6 x 6 as indicated by the options listed above.

Since this element is based on the BCIZ element, the derivation is initially performed using the deformational degrees of freedom. The following transformation converts the deformational degrees of freedom into nodal degrees of freedom.

$$\{\delta^*\} = [T^*]\{q\} \quad , \tag{6-29}$$

$$\begin{aligned} \lfloor q \rfloor = \lfloor & u_1,\ \theta_{ux_1},\ \theta_{uy_1},\ v_1,\ \theta_{vx_1},\ \theta_{vy_1},\ \omega_1,\ \theta_{\omega x_1},\ \theta_{\omega y_1}, \\ & u_2 \ldots \\ & u_3 \ldots \qquad\qquad\qquad\qquad \theta_{\omega y_3} \rfloor \end{aligned} \tag{6-30}$$

and the matrix $[T^*]$ is defined in the Apendix, Section 12.6. The stiffness matrix is thus transformed to a 27 x 27 matrix, involving the nodal degrees of freedom.

$$[K] = [T^*]^T [K^*][T^*] \tag{6-31}$$

7.0 QUADRILATERAL SHELL ELEMENT

The quadrilateral element follows the same basic philosophy as the triangle. It uses the strain displacement functions of Section 3.0, and again proceeds from a simple basic element to obtain an element with an HMN capability. In this case the isoparametric element of Ahmed, et al, Reference 3, with quadratic displacement functions, was used. This element has one midside node per side, uses the quadratic displacement forms for the membrane and normal displacements, and introduces transverse shear deformation which are also of quadratic form.

The HMN procedure basic approach is given in Section 5. The procedure differs somewhat in detail from that used for the triangle, due to inherent features of the element geometry and the nodal arrangement.

7.1 ASSUMED DISPLACEMENT SHAPES

The transverse displacement is assumed to be that of the AIZ element, Reference 3 . The displacement assumed for the parent element has the form, Figure 10

$$w = \sum_{1}^{8} F_i w_i \tag{7-1}$$

$$
\begin{aligned}
F_1 &= \tfrac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)\\
F_2 &= \tfrac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)\\
F_3 &= \tfrac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)\\
F_4 &= \tfrac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)\\
F_5 &= \tfrac{1}{4}(1+\xi)(1-\xi)(1-\eta)\\
F_6 &= \tfrac{1}{4}(1+\xi)(1-\xi)(1+\eta)\\
F_7 &= \tfrac{1}{4}(1+\eta)(1-\eta)(1+\xi)\\
F_8 &= \tfrac{1}{4}(1+\eta)(1-\eta)(1-\xi)
\end{aligned}
\tag{7-2}
$$

The functions F_i have the characteristic that they have the value of unity at node i and are zero at the other nodes.

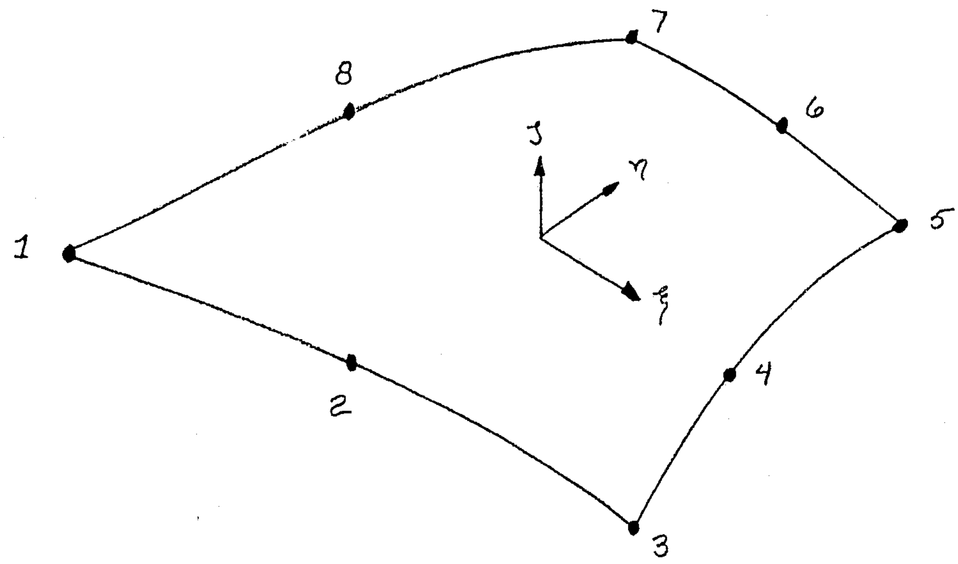
The membrane displacements u and v have the same form as the transverse displacement but they include the excess degrees of freedom and nodal rotations

$$u = \sum_{i=1}^{8} F_i u_i + \sum_{i=1}^{4} G_i a_i + \zeta \frac{t}{2} \sum_{i=1}^{8} F_i \theta_{y_i} \tag{7-3}$$

$$v = \sum_{i=1}^{8} F_i v_i + \sum_{i=1}^{4} \bar{G}_i b_i - \zeta \frac{t}{2} \sum_{1}^{8} F_i \theta_{x_i} \tag{7-4}$$

$$
\begin{aligned}
G_1 &= (\xi - \xi^3)(1-\eta) \\
G_2 &= (\xi - \xi^3)(1+\eta) \\
G_3 &= 2(\xi - \xi^3)(1-\eta^2) \\
G_4 &= (1-\xi^2)(1-\eta^2) \\
G_5 &= 2(1-\xi^2)(\eta-\eta^3) \\
G_6 &= \tfrac{1}{2}(\xi-1)(1-\eta)(1+\eta)\eta^2 \\
G_7 &= \tfrac{1}{2}(\xi+1)(1-\eta)(1+\eta)\eta^2 \\
G_8 &= \tfrac{1}{4}(\xi-1)(1-\eta)(1+\eta)\eta \\
G_9 &= \tfrac{1}{4}(\xi+1)(1-\eta)(1+\eta)\eta
\end{aligned}
\qquad (7\text{-}5)
$$

The functions $\overline{G}_i$ are identical to the G_i functions defined above except for $\overline{G}_i$ replace ξ by η and η by ξ in G_i.



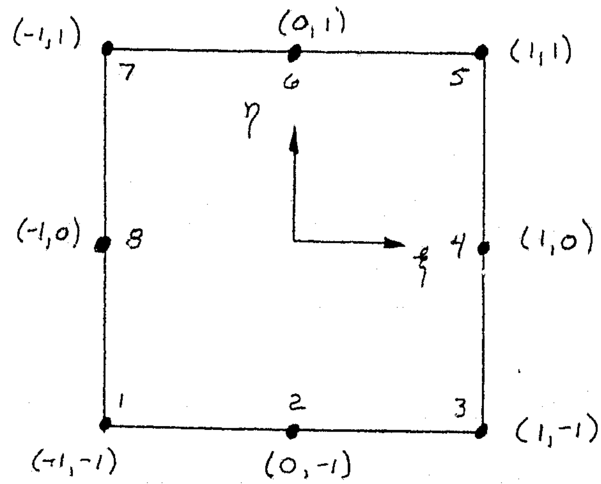


FIGURE 10 : NODAL ARRANGEMENT AND COORDINATE SYSTEM USED TO DEFINE DISPLACEMENT FUNCTIONS FOR QUADRILATERAL ELEMENT

The HMN functions are required to control third degree strains created by non-linear terms in the normal displacement. Hence, the HMN membrane displacements must contain terms through the fourth degree. Specifically, referring to the square parent element of the isoparametric family with local coordinates ξ and η parallel to the sides; (1) the $\tilde{u}$ displacements along lines parallel to the ξ axis must contain terms cubic in ξ and quadratic in η; (2) the $\tilde{u}$ displacements along lines parallel to the ξ axis must contain terms quadratic in ξ and cubic in η, which vanish on the sides of the element; (3) the $\tilde{v}$ displacements along lines parallel to the ξ axis must contain terms which are quartic in ξ and linear in η. These three types of functions are of course repeated for the alternate functions and lines parallel to the η axis. Figure 11 shows plots of these functions. Note that type (1) functions involve 3- $\tilde{u}$ and 3- $\tilde{v}$ unknowns; type (2) functions involve 2- $\tilde{u}$ and 2- $\tilde{v}$ unknowns, these being of the pillow function type; and type (3) functions involve 4- $\tilde{u}$ and 4- $\tilde{v}$ unknowns.

The functional forms are

Type (1)

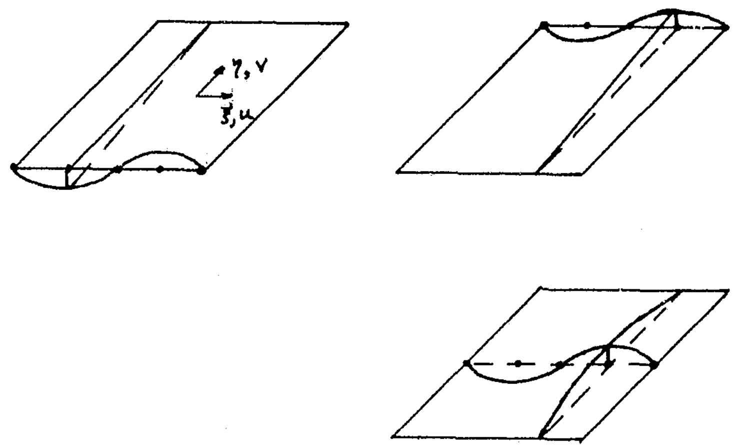
$$\begin{aligned} \tilde{u} &\rightarrow (1-\eta)(\xi-\xi^3)\,;\ (1+\eta)(\xi-\xi^3)\,;\ (1-\eta^2)(\xi-\xi^3) \\ \tilde{v} &\rightarrow (1-\xi)(\eta-\eta^3)\,;\ (1+\xi)(\eta-\eta^3)\,;\ (1-\xi^2)(\eta-\eta^3) \end{aligned} \tag{7-6}$$

Type (2)

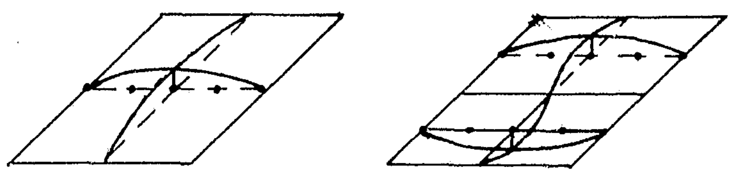
$$\begin{aligned} \tilde{u} &\rightarrow (1-\xi^2)(1-\eta^2)\,;\ (1-\xi^2)(\eta-\eta^3) \\ \tilde{v} &\rightarrow (1-\eta^2)(1-\xi^2)\,;\ (1-\eta^2)(\xi-\xi^3) \end{aligned} \tag{7-7}$$

Type (3)

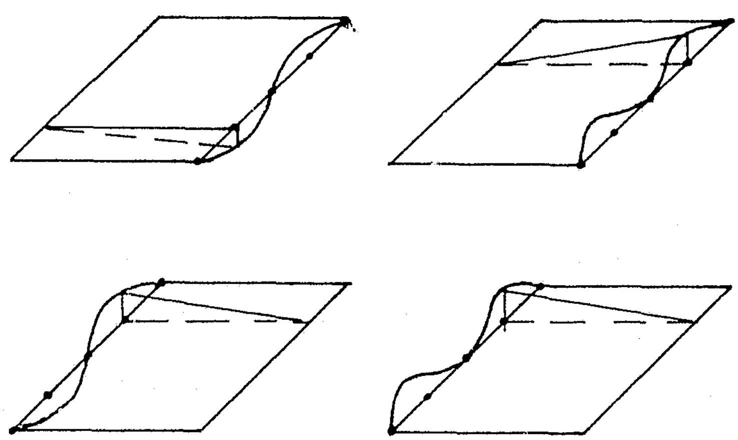
$$\begin{aligned} \tilde{u} \rightarrow\ & (\xi-1)\left[(1-\eta)(1+\eta)(\eta)^2\right];\ (\xi+1)\left[(1-\eta)(1+\eta)(\eta)^2\right] \\ & (\xi-1)\left[(1-\eta)(1+\eta)(\eta)\right];\ (\xi+1)\left[(1-\eta)(1+\eta)(\eta)\right] \\ \tilde{v} \rightarrow\ & (\eta-1)\left[(1-\xi)(1+\xi)(\xi)^2\right];\ (\eta+1)\left[(1-\xi)(1+\xi)(\xi)^2\right] \\ & (\eta-1)\left[(1-\xi)(1+\xi)(\xi)\right];\ (\eta+1)\left[(1-\xi)(1+\xi)(\xi)\right] \end{aligned} \tag{7-8}$$



TYPE 1 HMN $\tilde{u}$ FUNCTIONS



TYPE 2 HMN $\tilde{u}$ FUNCTIONS



TYPE 3 HMN $\tilde{u}$ FUNCTIONS

FIGURE 11 : HMN FUNCTIONS FOR QUADRILATERAL ELEMENT

HMN constraints sufficient to determine these functions consist of the following:

Constraints on

$\mathcal{E}_{\xi}$ on the lines $\eta = -1, 0, +1$

$\mathcal{E}_{\eta}$ on the lines $\xi = -1, 0, +1$

$\mathcal{E}_{\xi\eta}$ on the lines $\xi = \pm 1$ & $\eta = \pm 1$

plus minimum potential energy relative to the type (2) functions.

7.2 ELEMENT DERIVATION

The virtual work expressions presented in Section 4.0 are applicable to both the triangular and quadrilateral shell elements. The stiffness matrix was obtained in a general form in Equation (4-10), which is repeated here for clarity

$$K_{ij} = \int_A \left\{ \bar{B}_{si}\, DO_{s\ell}\, \bar{B}_{\ell j} + \bar{J}_{j\ell}\, \overline{GO}_{\ell r}\, \sigma_s^{\circ} A1_{sjk}\, \bar{J}_{kn} GO_{nj} \right\} DET|J|\, d\xi\, d\eta \qquad (7\text{-}9)$$

$$B_{ir} = (AO_{ij} + A1_{ijk}\, \bar{\bar{J}}_{KM}\, \bar{\theta}_m)\, \bar{J}_{j\ell}\, GO_{\ell r}$$

From Equation (4-4), the GO matrix has the form

$$\{\bar{\theta}\} = [GO]\{q^*\} \qquad (7\text{-}10)$$

$$\lfloor \bar{\theta} \rfloor = \lfloor u_\xi, u_\eta, v_\xi, v_\eta, \omega_\xi, \omega_\eta, \theta_x, \theta_y, \theta_{x,\xi}, \theta_{x,\eta}, \theta_{y,\xi}, \theta_{y,\eta} \rfloor \qquad (7\text{-}11)$$

$$\lfloor q^* \rfloor = \lfloor q \vdots \alpha_b \vdots \alpha_a \rfloor \qquad (7\text{-}12)$$

$$\begin{aligned} \lfloor q \rfloor = \lfloor u_1\ v_1\ \omega_1\ \theta_{x1}\ \theta_{y1}\quad & u_2\ v_2\ \omega_2\ \theta_{x2}\ \theta_{y2} \\ u_3 \ldots \quad & u_4 \ldots \\ u_5 \ldots \quad & u_6 \ldots \\ u_7 \ldots \quad & u_8\ v_8\ \omega_8\ \theta_{x8}\ \theta_{y8} \rfloor \end{aligned} \qquad (7\text{-}13)$$

$$\lfloor \alpha_b \rfloor = \lfloor a_4\ a_5\ a_8\ a_9\ b_4\ b_5\ b_8\ b_9 \rfloor$$

$$\lfloor \alpha_a \rfloor = \lfloor a_1\ a_2\ a_3\ b_6\ b_7\ a_7\ a_6\ b_1\ b_2\ b_3 \rfloor \qquad (7\text{-}14)$$

The explicit form of the GO matrix is given in the Appendix, Section 12.7.

However, since the displacement functions are prescribed for the parent element in the ξ, η coordinate system, an additional Jacobian type transformation is required to obtain the displacement gradients in the shell coordinate system.

$$\{\theta\} = [\bar{J}]\{\bar{\theta}\} \tag{7-15}$$

$$\lfloor \theta \rfloor = \lfloor u_x, u_y, v_x, v_y, \omega_x, \omega_y, \theta_x, \theta_y, \theta_{x,x}, \theta_{x,y}, \theta_{y,x}, \theta_{y,y} \rfloor \tag{7-16}$$

The transormation matrix $[\bar{J}]$ is given in the Appendix, Section 12.8.

The A0 and A1 matrices for the quadrilateral can be written directly from Equations as

$$\Delta E_i = (A0_{ij} + A1_{ijk}\,\theta_k)\,\Delta\theta_j \tag{7-17}$$

where

$$\lfloor \Delta E \rfloor = \lfloor \Delta E_x, \Delta E_y, \Delta E_{xy}, \Delta E_{xz}, \Delta E_{yz}, \Delta K_{xx}, \Delta K_{yy}, \Delta K_{xy} \rfloor \tag{7-18}$$

Explicit forms for A0 and A1 are given in the Appendix, Section 12.9. The curvatures (K_{xx}, K_{yy}, K_{xy}) are related to the bending strains by $E_{xx}^{(\text{Bending})} = z\,K_{xx}$, etc.

Finally, the stress-strain relation is given in Equation (4-7) as

$$\sigma_i = \sigma_i^{\circ} + DO_{i\ell}\,\Delta\mathcal{E}_\ell \tag{7-19}$$

where

$$\lfloor \sigma \rfloor = \lfloor N_x, N_y, N_{xy}, N_{xz}, N_{yz}, M_x, M_y, M_{xy} \rfloor$$
$$\lfloor \sigma^{\circ} \rfloor = \lfloor N_x^{\circ}, N_y^{\circ}, N_{xy}^{\circ}, N_{xz}^{\circ}, N_{yz}^{\circ}, M_x^{\circ}, M_y^{\circ}, M_{xy}^{\circ} \rfloor \tag{7-20}$$

and the matrix of material constants [DO] is given in the Appendix, Section 12.10.

With these matrices, the gross stiffness matrix of Equation (7-9) can be generated. This matrix is of order 58 by 58 involving the deformational degrees of freedom $\lfloor \Delta q \rfloor$, the degrees of freedom to be eliminated through minimum energy reduction $\lfloor \Delta\alpha_b \rfloor$,

and the degrees of freedom to be eliminated through HMN type constraints $\lfloor \Delta\alpha_a \rfloor$. These latter constraints have the form

$$[H]\{\Delta\alpha_a\} = [HO]\{\Delta\omega\} + [H1(\omega)]\{\Delta\omega\} \tag{7-21}$$

$$\lfloor \Delta\omega \rfloor = \lfloor \Delta\omega_1, \Delta\omega_2, \Delta\omega_3, \Delta\omega_4, \Delta\omega_5, \Delta\omega_6, \Delta\omega_7, \Delta\omega_8 \rfloor \tag{7-22}$$

$$H1_{ij} = \overline{H1}_{ijk}\,\omega_k \tag{7-23}$$

and the matrices $[H]$, $[HO]$, and $[\overline{H1}]$ are developed using Table 3. As indicated in Section 5.0, these equations can be put in the form

$$\begin{Bmatrix} \Delta q \\ \Delta\alpha_b \\ \Delta\alpha_a \end{Bmatrix} = [\tilde{\tilde{C}}] \begin{Bmatrix} \Delta q \\ \Delta\alpha_b \end{Bmatrix} \tag{7-24}$$

and the HMN constrained stiffness matrix is then

$$[\tilde{K}] = [\tilde{\tilde{C}}]^T [K] [\tilde{\tilde{C}}] \tag{7-25}$$

The constraint equations are obtained by the procedure developed in Section 5.0. As applied to the quadrilateral, the incremental membrane strains can be written

$$\Delta E_\xi = \frac{\partial u}{\partial \xi} + \frac{\partial \omega^\circ}{\partial \xi}\frac{\partial \Delta\omega}{\partial \xi} + \frac{\partial \omega}{\partial \xi}\frac{\partial \Delta\omega}{\partial \xi} \tag{7-26}$$

$$\Delta E_\eta = \frac{\partial \Delta v}{\partial \eta} + \frac{\partial \omega^\circ}{\partial \eta}\frac{\partial \Delta\omega}{\partial \eta} + \frac{\partial \omega}{\partial \eta}\frac{\partial \Delta\omega}{\partial \eta} \tag{7-27}$$

$$\Delta E_{\xi\eta} = \frac{\partial \Delta u}{\partial \eta} + \frac{\partial \Delta v}{\partial \xi} + \frac{\partial \omega^\circ}{\partial \xi}\frac{\partial \Delta\omega}{\partial \eta} + \frac{\partial \omega^\circ}{\partial \eta}\frac{\partial \omega}{\partial \xi} + \frac{\partial \omega}{\partial \xi}\frac{\partial \Delta\omega}{\partial \eta} + \frac{\partial \omega}{\partial \eta}\frac{\partial \Delta\omega}{\partial \xi} \tag{7-28}$$

Table 3 : BASIS OF QUADRILATERAL ELEMENT HMN EQUATIONS

STRAIN	EQUATE TO ZERO THE COEFFICIENT OF:	LOCATION
$\Delta \varepsilon_\xi$	$-6\xi^2$	$\eta = 1$
$\Delta \varepsilon_\xi$	$-6\xi^2$	$\eta = 0$
$\Delta \varepsilon_\xi$	$-6\xi^2$	$\eta = -1$
$\Delta \varepsilon_\xi$	$-4\xi^2$	$\eta = 1$
$\Delta \varepsilon_\xi$	$-4\xi^2$	$\eta = -1$
$\Delta \varepsilon_\xi$	$-4\xi^2$	$\xi = 1$
$\Delta \varepsilon_\xi$	$-4\xi^2$	$\xi = 1$
$\Delta \varepsilon_\xi$	$-6\xi^2$	$\xi = -1$
$\Delta \varepsilon_\xi$	$-6\xi^2$	$\xi = 0$
$\Delta \varepsilon_\xi$	$-6\xi^2$	$\xi = -1$

Ten HMN type equations can be written by evaluating these strains along various edges or locations within the quadrilateral and equating to zero the multipliers of terms in the strains. The basis for these equations is summarized in Table 3. With the displacement functions defined in Section 7.1, an algebraic expansion of Equations (7-26-28), yields the HMN constraint equations in the form of Equation (7-21). These equations are presented in the Appendix, Section 12.11. Application of these constraints yields the stiffness matrix in the form of Equation (7-25).

The remaining degrees of freedom, $[\alpha_b]$, are then eliminated through minimum energy reduction.

$$[K^*] = [\tilde{K}_{qq}] - [\tilde{K}_{qb}][\tilde{K}_{bb}]^{-1}[\tilde{K}_{bq}]$$

where the order of $[K^*]$ is 40 by 40, five degrees of freedom at each of the eight nodes.

8.0 COORDINATE UPDATING TRANSFORMATION

A special type of updating is used in the nonlinear solution procedure. This updating essentially uses convected, or material, coordinates, pertaining to the element start-of-step orientation, to set up an updated Lagrangian formulation of the incremental equations. The updating requires a special transformation to permit the entire solution process to be purely Lagrangian in character. For each new incremental solution the accumulated total and the incremental forces and displacements are referred to the end-of-step element base plane coordinate system of the previous step. The updated end-of-step element coordinate systems are determined for the iteratively converged shape of the structure at the end of the particular increment in question. Strains and stresses are computed with reference to the updated systems permitting a determination of total strain and stress values without the necessity of summing increments. Updated coordinate systems are also used as temporary coordinate systems to facilitate the iterative calculations. In this case the purpose is to obtain an exact strain/stress state on which to base the calculation of residual loads and the iterative equilibrium corrections. The sequence of computations is sketched briefly below:

- Determine end of step element baseplane coordinate system (converged).
- Compute stiffness matrix, loads, etc., referred to this as a new start-of-step system.
- Compute incremental displacements, referred to above.
- Determine temporary element end-of-step coordinate system, corresponding to computed incremental displacements.
    - Transform displacements to the temporary coordinate system*
    - Calculate strain and stress (total referred to the (temporary) end-of-step coordinate system**)
    - Determine residual loads and check convergence
    - Correct incremental displacements, to eliminate residual loads (uses start-of-step stiffness matrix)

---

* This is a special transformation, to be derived in this section.

** This is not by summing increments; it is a direct total calculation. The strains and stresses are of the Lagrangian type.

- o Combine correction displacements with prior displacement increment and repeat determination of temporary end-of-step coordinate system, strain, stress, and residual load calculations, etc.
- o Identify the last temporary end-of-step coordinate system as the final, converged end-of-step element coordinate system.

At this point the entire calculation passes on to the next loading increment. The remaining small residual loads are carried on into the next step to avoid accumulating error. A more detailed description of the solution procedure is given in the next section. The above brief listing serves to explain the role of the special displacement updating transformation which is employed to compute total strains. The derivation of this transformation is the primary purpose of this section.

Consider Figure 12. An element is shown referred to three rectangular cartesian coordinate systems: the initial one x_o^i for the undeformed structure; a coordinate system $\bar{x}^i$ whose $\bar{x}^1$, $\bar{x}^2$ coordinate plane is parallel to the element in its deformed state; and a coordinate system $\tilde{x}^i$ whose $\tilde{x}^1$, $\tilde{x}^2$ plane is parallel to the element after a further increment of displacement has occurred. For simplicity the figure is drawn in only two dimensions. The derivation given, however, includes all three components of displacements and coordinates. The original coordinates x_o^1 are a cartesian version of the material coordinates. In the cartesian sense, x_o^i can be considered to be convected with the deformation. Thus each of $\bar{x}^i$ and $\tilde{x}^i$ have companion colinear x_o^i systems. As the deformation progresses, the material coordinates are deformed and rotated but still partially coincide in direction with the $\bar{x}^i$ and $\tilde{x}^i$ coordinates, as follows: the convected x_o^1 axis coincides in direction with the $\bar{x}^1$ and $\tilde{x}^1$ axes and the convected $x_o^1 - x_o^2$ plane coincides with the $\bar{x}^1$, $\bar{x}^2$ and $\tilde{x}^1$, $\tilde{x}^2$ planes. In addition, the side of the element containing nodes 1 and 2 coincides in direction with the convected x_o^1, and $\bar{x}^1$, and $\tilde{x}^1$ axes, and node 1 is at the origin of all coordinate systems. Thus, the coordinate systems $\bar{x}^1$ and $\tilde{x}^i$ are the rectangular cartesian counterparts of the element convected coordinate system, with the conventions of coincidence of the origin, the number one coordinate line, and the 1-2 plane. Note that the metric of $\bar{x}^i$ and $\tilde{x}^i$ contains only zero and unity, while the metric of x_o^i does not have unit values after deformation has occurred.

Displacements, both total and incremental, used in each step, are referred to the current coordinate systems. Hence for the increment whose start-of-step system is $\bar{x}^i$, the displacements $\bar{u}^i(x_o^j)$, $\Delta\bar{u}^i(x_o^j)$, and $\bar{u}^i(x_o^j) + \Delta\bar{u}^i(x_o^j)$ are components in the directions of the $\bar{x}^i$ axes. The total strains and the incremental strains of the Lagrangian type, referred to the $\bar{x}^i$ system, viewed as the convected x_o^i system, are derivable from these displacements provided that they represent, in the appropriate coordinate system, precisely the deflections from an undistorted shape. At the end of the step, the $\bar{u}^i + \Delta\bar{u}^i$ are transformed to refer them to the end-of-step $\tilde{x}^i$ system, again as deflections from an undistorted shape. The derivation of a transformation of $\bar{u}^i + \Delta\bar{u}^i$ from the $\bar{x}^i$ system to the end-of-step $\tilde{x}^i$ system such that the resulting displacements $\tilde{u}^i$ adhere to this definition is given below.

The undeformed shape in the x_o^i coordinate system is described by the position vector (Figure 12)

$$\vec{P}_o = x_o^i \vec{G}_i \tag{8-1}$$

where here $\vec{G}_i$ are the base (unit) vectors of the x_o^i system. Here x_o^i are the coordinates of points on the undeformed shape of the element. If the undeformed element is placed into the $\bar{x}^i$ system (shown dotted in the figure), its position vector relative to the origin of the $\bar{x}^i$ system has the same form,

$$\vec{\bar{P}}_o = x_o^i \vec{g}_i \tag{8-2}$$

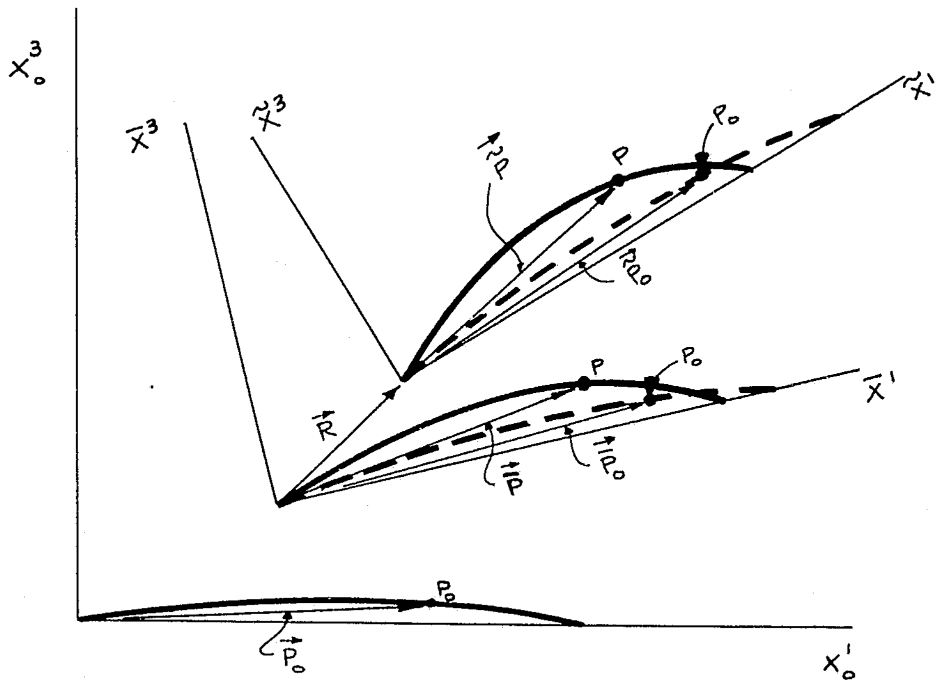


FIGURE 12 : COORDINATE SYSTEMS AND DEFINITIONS FOR COORDINATE SYSTEM UPDATING PROCEDURE

where here $\vec{g}_i$ are the (unit) base vectors of the $\bar{x}^i$ system. The actual shape of the deformed element is given by

$$\vec{\bar{P}} = (x_o^i + \bar{u}^i)\ \vec{g}_i \tag{8-3}$$

where here the $\bar{u}^i$ are implicitly defined by this equation to be the deviation of the deformed element shape from the undeformed shape, referred to the $\bar{x}^i$ system. After the incremental displacement,

$$\vec{\bar{P}} + \Delta\vec{\bar{P}} = (x_o^i + \bar{u}^i + \Delta\bar{u}^i)\ \vec{g}_i \tag{8-4}$$

This vector, referred to the $\bar{x}^i$ system, actually defines points on the element depicted by the solid lines in the figure and "attached" to the $\tilde{x}^i$ system. That is, the $\Delta\bar{u}^i$ are the displacements which carry the deformed element of the $\bar{x}^i$ system to the deformed element of the $\tilde{x}^i$ system; $\tilde{x}^i$ are the end-of-step coordinates of the step for which $\bar{x}^i$ are the start of step coordinates.

The position vector of a generic point of the element in the $\tilde{x}^i$ system is given by

$$\vec{\tilde{P}} = (x_o^i + \tilde{u}^i)\ \vec{h}_i \tag{8-5}$$

where $\vec{h}_i$ are the (unit) base vectors of the $\tilde{x}^i$ system. The implicit definition of $\tilde{u}^i$ as the deviation from an undeformed shape is again apparent. Note that $\tilde{u}^i$ defines the deformation condition which corresponds to $\bar{u}^i + \Delta\bar{u}^i$.

Let $\vec{R}$ be the displacement of the origin from $\bar{x}^i$ to $\tilde{x}^i$. Then $\vec{\bar{P}} + \Delta\vec{\bar{P}} = \vec{R} + \vec{\tilde{P}}$, i.e.,

$$\left[x_o^i + (\bar{u}^i + \Delta\bar{u}^i)\right]\vec{g}_i = \vec{R} + (x_o^i + \tilde{u}^i)\vec{h}_i \tag{8-6}$$

Defining $\vec{h}_i \bullet \vec{g}_j = \lambda_{ij}$, (8-7)

$$\left[x_o^j + (\bar{u}^j + \Delta\bar{u}^j)\right]\lambda_{ij} = \vec{R}\bullet\vec{h}_i + x_o^i + \tilde{u}^i \tag{8-8}$$

This gives the desired transformation and its derivative with respect to the material coordinates.

$$\tilde{u}^i = (\bar{u}^j + \Delta\bar{u}^j)\lambda_{ij} + x_o^j\lambda_{ij} - x_o^i - \vec{R}\bullet\vec{h}_i \tag{8-9}$$

$$\frac{\partial\tilde{u}^i}{\partial x_o^k} = \left(\frac{\partial\bar{u}^j}{\partial x_o^k} + \frac{\partial\Delta\bar{u}^j}{\partial x_o^k}\right)\lambda_{ij} + \lambda_{ik} - \delta_{ik} \tag{8-10}$$

In the above, distinction between contravariant and covariant components has not been made, since the coordinate systems are cartesian.

These equations transform the end of step deformations referred to the $\bar{x}^i$ system to the corresponding start of step deformation referred to the $\tilde{x}^i$ system. This transformation is used for updating the derivative freedoms of the element in order to define the total deformation state of the elements both at the end of the step and equivalently, for the start of the next step. The displacement freedoms can be transformed using the above equations, or, alternatively, they can be obtained by simply subtracting the deformed nodal coordinates from the undeformed ones, in the $\tilde{x}^i$ system, at the end of the step. Note that in the case of the triangular element the above are "total" derivative values rather than values of the BCIZ type which omit the simplex type terms. Thus, the BCIZ transformation is required to be applied to the above types of derivatives in order to compute strains in the element using the standard BCIZ formulas.

9.0 SOLUTION PROCEDURE

Nonlinear solution procedures and the formulations of nonlinear finite elements are interrelated. Frequently derivation details are specifically tailored to planned solution procedures. In addition, precisely how the element equations are used, including updating, transformation, summing increments, etc., in the solution procedure has important implications regarding the performance of the element and the validity of the element derivation steps. The elements developed in this research have close interrelationships between derivation details and solution procedure steps. This section discusses the solution procedure and related element derivation details.

The solution procedure is a modified incremental Lagrangian type. Each incremental step is subjected to iterative equilibrium corrections which involve the calculation and reversed application of residual loads. To accomplish this, element total strains are computed directly (not by summing increments), corresponding to the total displacement state. The total strain calculation employs an element coordinate system updating which involves the special transformation described in Section 8.0 The overall procedure can be classified as either a total or an incremental Lagrangian type, with updating.

The steps in the solution procedure are described below:

1. The stiffness matrices are generated based on the previous end-of-step displacements, stress state, and updated element coordinate plane. The element stiffness matrices are transformed to the solution coordinate systems, using the locations of these systems for the end of the previous step. The element matrices are then merged to form the overall stiffness matrix, referred to the solution coordinate systems. (This is not the global system.) In iterations pertaining to a given load increment, the stiffness matrix and related transformation matrices are not changed.

2. The load increment, including the remaining residual load from the end of the previous step, is obtained and transformed to current solution coordinate systems.

3. The incremental displacement $\Delta\hat{q}^{(j)}$ is calculated. This is the first estimate, uniterated, of the displacement increment for the current load increment.

4. The $\Delta\hat{q}^{(j)}$ above are transformed into the element baseplane coordinate system

$$\left\{\Delta q^{(j)}\right\} = \left[\Lambda^{(j)}\right]\left\{\Delta\hat{q}^{(j)}\right\}$$

5. For the triangle only, for use only in a later calculation of the minimum potential energy freedom increments, the above $\Delta q^{(j)}$ are transformed to the "deformational" incremental displacements. These are the increments used in the BCIZ element derivation.

$$\left\{\Delta\delta^{*(j)}\right\} = \left[T^{*}\right]\left\{\Delta q^{(j)}\right\}$$

For the quadrilateral this would be an identity transformation.

6. The element incremental degrees of freedom are summed into the previous totals to form new total element freedom values. This uses the values $\delta^{*(j-1)}_{SIM}$ from the previous incremental calculation, as defined in Steps 14 and 15.

$$\delta^{*(j)}_{SIM} = \delta^{*(j-1)}_{SIM} + \Delta q^{(j)}$$

The sum obtained in this way contains the current estimate of the displacements of the element, relative to its undeformed shape, and referred to the current start-of-step element baseplane coordinate systems. The displacement derivative freedoms obtained are total rather than deformational. That is, for example, $\frac{du}{dx}$ includes not only the BCIZ deformational contribution, but also the contribution due to the nodal deflections. Since the latter are the "simplex" values, we have used the subscript "SIM" to denote the fact that the

derivative freedoms in $\delta_{SIM}^{*(j)}$ contain the simplex contributions.

7. The incremental deflections are transformed to the global coordinate system and summed

$$\{\Delta U^{(j)}\} = [\hat{T}^{(j)}]\{\Delta\hat{u}^{(j)}\}$$

$$\{U^{(j)}\} = \{U^{(j-1)}\} + \{\Delta U^{(j)}\}$$

Here the $\Delta\hat{u}^{(j)}$ are the nodal deflections contained in the displacement $\Delta\hat{q}^{(j)}$ from Step 3.

8. The element geometry and location in the global coordinate system is updated in order to update the element baseplane coordinate system.

$$\{X^{(j)}\} = \{X^{(j-1)}\} + \{\Delta U^{(j)}\}$$

Note that this updating is not a generic element updating. See Section 8.0

9. A new matrix of direction cosines relating the updated element baseplane coordinate system to the global system is calculated. These are called

$$[T^{(j)}]$$

$[T^{(j)}]$ is defined such that nodes 1 and 2 are on the updated x axis and either (for the triangle) nodes 1, 2 and 3 are in the xy plane, or (for the quadrilateral), the nodes are displaced from the xy plane such that their RMS departure is minimized.

10. The current nodal coordinates referred to the updated baseplane coordinate system are calculated. For the triangle, the set of values includes only x_2, x_3 and y_3, due to the definition of $[T^{(j)}]$ given above. For the quadrilateral it includes all x_i, y_i except y_4, y_8, plus the z coordinate for all nodes. The equation is

$$[x^{(j)}] = [T^{(j)}][\bar{X}^{(j)}]$$

Here, $[\bar{X}^{(j)}]$ contains the quantities $X_{mn}^{(j)}$ where $X_{mn}^{(j)} = X_m^{(j)} - X_n^{(j)}$ refers to the X, Y, and Z global coordinates.

The result of this calculation is that the nodal coordinates are known for the current state of deformation in the current end-of-step updated element baseplane coordinate system. These data permit calculation of nodal deflections relative to the undeformed element referred to this same coordinate system. The updated element coordinate system is used essentially as a material, or convected, coordinate system.

11. The element total nodal translations are next calculated from the equation

$$\left\{ \tilde{u}^{(j)} \right\} = \left\{ x^{(j)} \right\} - \left\{ x^{(o)} \right\}$$

The notation $\tilde{(\ )}$ denotes the fact that these translations are referred to the current end-of-step element baseplane coordinate system. The $\tilde{u}^{(j)}$ include the in-plane defections only. For the triangle, the normal-to-the-updated-baseplane deflections are equal to zero. For the quadrilateral, the normal deflection are set equal to the changes in the normal coordinates determined by the RMS fit of the four nodes to the updated element baseplane.

12. The incremental minimum energy freedoms are computed by back substitution using the same constraint matrices used originally to constrain the stiffness matrix referred to in Step 1. This calculation effectively uses the tangent stiffness of the element for the current increment referred to its start-of-step coordinate plane. The computed quantities are

$$\left\{ \Delta\alpha_b^{(j)} \right\} = \left\{ \Delta\alpha_1^{(j)} \right\} , \text{ or } \left\{ \begin{matrix} \Delta\alpha_1^{(j)} \\ \Delta\alpha_2^{(j)} \end{matrix} \right\} , \text{ or } \left\{ \begin{matrix} \Delta\alpha_1^{(j)} \\ \Delta\alpha_2^{(j)} \\ \Delta\alpha_3^{(j)} \end{matrix} \right\}$$

depending on the particular HMN constraint options employed.

13. The minimum energy freedoms are next summed to obtain totals

$$\left\{ \alpha_b^{(j)} \right\} = \left\{ \alpha_b^{(j-1)} \right\} + \left\{ \Delta\alpha_b^{(j)} \right\}$$

These quantities are not subjected to any geometrical transformation type of updating.

14. The displacement derivative freedoms are next transformed to the end-of-step element baseplane coordinate system using the $\dot{R}$ transformation

of Section 8. The equation is

$$\left\{ \tilde{\delta}^{*(j)}_{SIM} \right\} = \left[\dot{R1} \right] \left\{ \delta^{*(j)}_{SIM} \right\} + \left\{ \dot{R2} \right\}$$

where the notation $\tilde{(\ )}$ denotes displacements and differentiations referred to the end-of-step element baseplane coordinate system and the subscript "SIM" again notes that the derivatives obtained are total rather than BCIZ deformational quantities; they contain the simplex contributions. The transformation is applied selectively to the derivative freedoms of $\{\delta^{*(j)}_{SIM}\}$, since the nodal translations themselves were computed in Step 11.

15. The nodal translations from Step 11, which are also referred to the end-of-step system, are entered into the $\{\tilde{\delta}^{*(j)}_{SIM}\}$ vector, replacing the corresponding translations term by term. At this point, $\{\tilde{\delta}^{*(j)}_{SIM}\}$ describes completely the element deformation state, relative to the undeformed element shape, and referred to the end-of-step element baseplane coordinate system. Note that the resulting $\{\tilde{\delta}^{*(j)}_{SIM}\}$ is the vector $\{\delta^{*(j-1)}_{SIM}\}$ to be used in Step 6 in performing the next incremental calculation. $\{\tilde{\delta}^{*(j+1)}_{SIM}\}$ is saved for this purpose.

16. The BCIZ transformation $[T^*]$ is applied to $\tilde{\delta}^{*(j-1)}_{SIM}$ to remove the simplex contributions to the derivatives.

$$\left\{ \tilde{\delta}^{*(j)} \right\} = \left[T^* \right] \left\{ \tilde{\delta}^{*(j)}_{SIM} \right\}$$

This prepares the vector $\{\tilde{\delta}^{*(j)}\}$ for use in total strain calculation in the end-of-step coordinate system. The resulting strains will be total Lagrangian values.

17. The total HMN freedoms $\{\alpha_a^{(j)}\}$ are next calculated. These freedoms are determined by constraining the polynomial forms of the strains along the element sides. The equations governing the total HMN freedoms, in the present form of the computer program, are of the form

$$\left\{ \alpha_a^{(j)} \right\} = \left[CO^{(j)} \right] \left\{ \begin{matrix} \tilde{\theta}^{*(j)}_{w} \\ \alpha_b^{(j)} \end{matrix} \right\} + \frac{1}{2} \left[C1^{(j)}_{mnp} \, \tilde{\theta}^{*(j-1)}_{w_p} \right] \left\{ \tilde{\theta}^{*(j)}_{w} \right\}$$

for the triangle, and

$$\left\{\alpha_a^{(j)}\right\} = \left[co^{(j)}\right]\left\{\tilde{w}^{(j)}\right\} + \frac{1}{2}\left[c1_{mnp}^{(j)}\,\tilde{\tilde{\omega}}_p^{(j-1)}\right]\left\{\tilde{w}^{(j)}\right\}$$

for the quadrilateral. Here $\{\alpha_a\}$ are the HMN function amplitudes and $\{\alpha_b^{(j)}\}$ are the minimum potential energy parameters. $\{\tilde{\theta}_w^{*(j-1)}\}$ and $\{\tilde{\theta}_w^{*(j)}\}$ are nodal values of the derivatives of $\{\tilde{w}^{*(j-1)}\}$ and $\{\tilde{w}^{*(j)}\}$, respectively, with respect to the current x and y element baseplane coordinates. $\{\tilde{\theta}_w^{*(j-1)}\}$ and $\{\tilde{\theta}_w^{*(j)}\}$ are contained in $\{\tilde{\delta}^{*(j-1)}\}$ and $\{\tilde{\delta}^{*(j)}\}$ where these vectors are as defined in Steps 14 and 15. $\{\tilde{\delta}^{*(j-1)}\}$ refers here to the deformations at the end of the (j-1)st step, referred to the end-of-step baseplane of that step. $\{\tilde{\delta}^{*(j)}\}$ refers here to the deformations at the end of the jth step, referred to the end of step baseplane of that step. Equations similar to the above are used in Step 1 to constrain the stiffness matrix for the incremental calculation. From these computations the matrices $[c0^{(j)}]$ and $[c1_{mnp}^{(j)}\tilde{\theta}_{w_p}^{*(j-1)}]$ are saved. These saved matrices are used to compute the total HMN freedoms at the end of the jth increment.

The use of $\{\tilde{\theta}_w^{*(j-1)}\}$ and $\{\tilde{\theta}_w^{*(j)}\}$ together is an approximation which has been found of small effect. It can be made as exact as desired by the use of zero load increments. The exact equation, of course, would result from reforming the matrix $[c1_{mnp}^{(j)}\tilde{\theta}_w^{*(j)}]$ using the current $\tilde{\theta}_w^{*(j)}$ referred to the end-of-step baseplane coordinate system.

For the triangle,

$$\left\{\alpha_a^{(j)}\right\} = \left\{\begin{matrix}\alpha_2^{(j)}\\ \alpha_3^{(j)}\end{matrix}\right\} \quad \text{or} \quad \left\{\alpha_3^{(j)}\right\} \quad \text{or} \quad \left\{\text{null}\right\}$$

according to which HMN option is chosen.

It is noted that the above HMN procedure, particularly if employed with occasional zero load increments, determines the HMN freedoms to control the total Lagrangian strain, obtained by reference to a current, updated, element baseplane. Thus, the HMN freedoms are not subject to accumulated error.

18. At this point the current totals of all freedoms are available, referred to the end-of-step element coordinate system. These totals consist of two sets, as follows:

$$\left\{ \tilde{q}^{*(j)}_{SIM} \right\} = \left\{ \begin{matrix} \tilde{\delta}^{*(j)}_{SIM} \\ \alpha^{(j)}_{a} \\ \alpha^{(j)}_{b} \end{matrix} \right\}$$

and

$$\left\{ \tilde{q}^{*(j)} \right\} = \left\{ \begin{matrix} \tilde{\delta}^{*(j)} \\ \alpha^{(j)}_{a} \\ \alpha^{(j)}_{b} \end{matrix} \right\}$$

The former takes $\left\{\tilde{\delta}^{*(j)}_{SIM}\right\}$ from Step 14 and is used for summing to obtain totals for subsequent increments, as described in Step 6 and further explained in Steps 14 and 15. The latter takes $\left\{\tilde{\delta}^{*(j)}\right\}$ from Step 16 and is used in the jth increment for HMN function calculation, just described, and also for strain and stress calculation. The latter calculations permit the determination of residual (error) loadings, and hence, iteration of the solution to improve overall equilibrium.

Steps 19 through 22 are calculations performed for each integration point of the element, leading to a virtual work integration performed in Step 23.

19. The matrix of displacement gradients is calculated at all integration points

$$\tilde{\theta}^{*(j)}_{m} = GO_{mn}\, \tilde{q}^{*(j)}_{n}$$

20. The strains are calculated

$$E_m^{(j)} = \left(AO_{mn} + \frac{1}{2} A1_{mnp} \tilde{\theta}_p^{*(j)} \right) \tilde{\theta}_n^{*(j)}$$

These are the true Lagrangian strains, referred to original material "fibers" provided the membrane deformations are small.

21. The stresses are calculated

$$\sigma_m^{(j)} = \sigma_m^{(o)} + DO_{mn} E_n^{(j)}$$

22. Quantities are next calculated which will provide virtual strains at each integration point and lead to the virtual work integration

$$\tilde{A}_{mn}^{(j)} = AO_{mn} + A1_{mnp} \tilde{\theta}_p^{*(j)}$$

$$\tilde{Q}_m^{(j)} = \sigma_n^{(j)} \tilde{A}_{nm}^{(j)}$$

$$\overline{\overline{\overline{Q}}}_m^{(j)} = (WT)\, GO_{pm} \tilde{Q}_p^{(j)}$$

Here (WT) are the weighting numbers for the particular integration scheme employed. The stresses $\sigma_n^{(j)}$ are taken from Step 21 and describe the current end-of-step state of the element. The subscript in $\tilde{Q}_m^{(j)}$ denotes the particular element freedom corresponding to which the virtual work of the $\sigma_n^{(j)}$ is sought. The result of summing the $\overline{\overline{\overline{Q}}}_m^{(j)}$ over all integration points is the total internal virtual work corresponding to a unit virtual increment in the m^{th} freedom, and provides the current generalized load on the element corresponding to that freedom, referred to the end-of-step coordinate system.

23. The current generalized loads on the element are computed by summing over the integration points

$$\overline{\overline{Q}}_m^{(j)} = \sum_{\substack{\text{INTEGRATION} \\ \text{POINTS}}} \overline{\overline{\overline{Q}}}_m^{(j)}$$

24. The constraints associated with the HMN and minimum potential energy conditions and, for the triangle, the BCIZ transformation, must be imposed on the $\overline{\overline{Q}}_m^{(j)}$ in order to obtain generalized forces which correspond to applied loadings in meaning and ordering. The equations are, respectively

$$\{\dot{Q}^{(j)}\} = [\tilde{\tilde{c}}]^T \{\bar{\bar{Q}}^{(j)}\}$$

$$\{Q^{*(j)}\} = \{\dot{Q}^{(j)}_{\delta^*}\} - [K^{(j)}_{\delta^* b}][K^{(j)}_{bb}]^{-1}\{\dot{Q}^{(j)}_b\}$$

$$\{\tilde{Q}^{(j)}\} = [T^{*(j)}]^T \{Q^{*(j)}\}$$

Definitions of terms in the above equations are given in earlier, related sections of this document. The $\{\tilde{Q}^{(j)}\}$ are the element nodal forces referred to the end-of-step element coordinate system.

25. The element forces are next transformed into the current, end-of-step solution coordinate system, using the updated matrix $[\tilde{\Lambda}^{(j)}]$ (see Step 4).

$$\{\hat{Q}^{(j)}\} = [\tilde{\Lambda}^{(j)}]\{\tilde{Q}^{(j)}\}$$

26. The $\{\hat{Q}^{(j)}\}$ for all elements are merged into a single vector, each member of which refers to the appropriate end-of-step, nodal, solution coordinate systems, and the residual loads are computed by subraction from the applied loads, also transformed to these coordinate systems,

$$\{\hat{R}^{(j)}\} = \{\hat{Q}^{(j)}_{\text{APPLIED}}\} - \{\hat{Q}^{(j)}\}$$

27. A convergence check is made to ascertain if the residual loads are suitably small, If so, the solution procedure returns to Step 1; if not, the steps below are carried out.

The iterative calculations described below are repetitions of the steps described previously, using different starting values of $\Delta\hat{q}^{(j)}$ as described below. In the current form of the computer program, the stiffness matrix for a given load increment is not updated for the iteration of that load increment. Of course, updating can be accomplished simply by the use of a zero load increment. It is again noted that the residuals remaining after convergence has been achieved for a given increment are always added into the loads for the next load increment.

28. The incremental load is set equal to the residual calculated in Step 26.

29. Step 3 is repeated, using stiffness matrix from Step 1. The results are correction values of $\{\Delta\hat{q}^{(j)}\}$. These are added directly to the previous results of Step 3, the result becoming an improved estimate of $\{\Delta\hat{q}^{(j)}\}$. Various schemes to accelerate convergence have been used here, mostly based on constraining the bending type corrective incremental displacements during the iteration. The constraints are applied on every other iterative step. The best procedure found is to use the membrane components of the stiffness matrix from Step 1 in every other iteration, and the full stiffness matrix in the alternate iterations. This cannot be recommended as a suitable procedure for general usage, however, as its success may be specific to the particular problems under study.

At this point Steps 4 through 27 are repeated exactly as described previously. The updating now is to an improved end-of-step element baseplane, and all transformations refer to this improved coordinate plane. In Step 17 the HMN calculation again makes use of the $\tilde{\theta}_w^{*(j-1)}$, the values from the previous load step, not those from the previous iteration of the current load step.

10.0 RESULTS AND CONCLUSIONS

Pilot computer programs for both the quadrilateral and triangular shell elements have been written. These programs can accommodate small assemblages of elements, and each includes a nonlinear solution procedure adapted to the element formulation.

Checkout of the computer programs has included successful compilation, a limited unsuccessful numerical application for the quadrilateral, and a detailed investigation of two different problem solutions for the triangle. The triangle has been used to work out solution procedure difficulties and to investigate the effect of the HMN formulation in nonlinear problems. This work has been successful, and it is felt that the value of the HMN application to two dimensional problems has been demonstrated. The following discussions of results pertain to the triangular element studies.

The problem solutions obtained serve to check the membrane only and the coupled membrane-bending behavior of the the triangular element. The various transformations, updating, etc., in the solution procedure are also checked by these problems. The problem solutions themselves are discussed in Section 10.1 and 10.2. The difficulties involved in the solution procedure are discussed below.

The solution procedure includes the calculation of residual loads and iteration to improve the structural equilibrium state. The basic solution proceeds on a stepwise linear basis. In general, the stepwise linearity gives rise to large residual forces due to the effects of the rotations. It was found that divergence of the solution occurred when the residuals exceeded certain magnitudes, in particular, magnitudes approaching structural instability loads. These limiting magnitudes were obtained with quite small load steps, with the consequence that practical solutions of large deflection problems required an excessive number of load steps. This difficulty has been resolved, and the solution procedure now performs well for load steps of essentially arbitrarily large size. Convergence is extremely rapid, and can in most cases be obtained in a single corrective iteration. The

current solution procedure uses a constrained, membrane only, stiffness matrix for alternate iterative corrections, and the total stiffness matrix for the complementary alternate iterations.

The quadrilateral and triangular elements both permit representation of either flat plate or curved shell structures. Curved shell checkout problems have not been attempted. It is noted, however, that the equations by which the elements represent nonlinearity due to developing rotations and curvatures for flat plate problems are the same as those which account for the effects of initial curvature. Thus, the flat plate checkout work has effectively dealt with the curved shell features to a significant extent.

10.1 TENT PROBLEM

The tent problem was designed to provide a simple check on the program flow and solution procedure. The loads and geometry are such that no bending occurs. The flat membrane stiffness, stresses, and most basic transformations and matrix operations are checked, however. The basic configuration, idealization and boundary conditions are shown in Figure 13. To reduce the problem size a plane of symmetry was assumed parallel to the Z axis and passing through nodes 2 and 4. This problem was successfully solved as shown by the results given in Figure 14. The seven step triangular element nonlinear solution is exactly the same as the theoretical solution.

10.2 BEAM PROBLEM

The beam problem is a simple example intended to exercise both bending and membrane behavior and to check the performance of the HMN procedure. The structure is a three element, initially flat plate shown in Figure 15. The figure also shows the local coordinate systems of the elements and the structural constraints imposed. As described in earlier sections, several options are programmed for the HMN procedure. The one used in this sample problem solution constrains the high degree membrane strains parallel to each element side, along the particular side in question. There are thus six HMN unknowns per element. The HMN functions provide displacements tangential

to a side, of quintic and quartic form in the side-length variable. All HMN displacements normal to the sides have been set at zero. All HMN interior functions have been retained and controlled by minimizing the total potential energy. This HMN option is option #1 as described in Section 6.2.

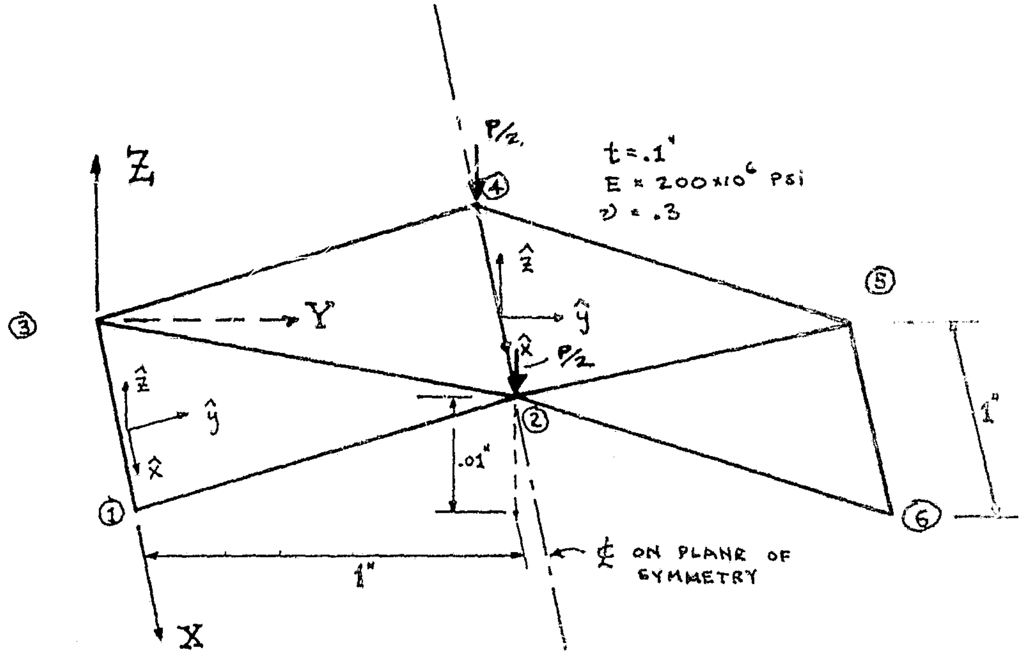
The overall stiffness in bending of the cantilevered beam of Figure 15 under end load is about 15% too high for the initial linear behavior. This is caused by the coarseness of the finite element modeling and the displacement function of the basic BCIZ elements, specifically the constrained BCIZ cubic displacement form. Figure 16 shows the end displacement versus load results, which are almost linear through the load range considered. The deviation from linearity seen in the figure is consistent with the effect of transforming the applied vertical load to the element solution coordinate system. The insert in the figure shows the distribution of bending moment on the edge of element #1 over the length of the beam. This distribution supports the conclusion that the element arrangement is too coarse for application to this problem with the basic BCIZ element. The total nodal moment reaction at the support end of the plate is in equilibrium with the applied load.

Figure 17 shows the spanwise stresses along side 1-3 of element number 1. The successive plots show the stresses at three load levels and both before and after a zero load step. It is recalled that the zero load step is used to allow the HMN procedure to improve in accuracy by implementing the most recently updated nonlinear strain values. For the lowest load level shown, the diagram on the left corresponds to zero values of the HMN function; no HMN participation has yet developed. The diagram on the right corresponds to essentially complete HMN effectiveness. Each of the two diagrams on the left for the higher load levels result from partial effectiveness of the HMN functions, while those on the right again correspond to essentially complete HMN effectiveness. It is seen in each case that the HMN procedure has reduced the stresses significantly. In addition, the forms of the residual stresses after updating are of relatively low order, apparently quadratic, of the BCIZ type, rather than of the high order form which would result if the HMN procedure were not used. In each

case the higher degree forms, which are seen to be predominantly cubic, occur only prior to the zero load step, when the HMN functions are not fully operative. Similar calculations have been made for sides 1-3 of elements #2 and #3. The stresses parallel to the side were computed before and after the zero load step. The results were essentially the same as those shown in Figure 17, with again the HMN functions reducing the stresses to very low levels. In this case the remaining residual stresses were distributed linearly over the length of the side. As would be expected, the stress parallel to the side has identical values in the adjoining elements. Finally, all components of the stresses in the interior of the element are very small also, of the same order as those parallel to the sides.

As another check on the levels of the membrane stresses, the membrane strain energy was computed and its magnitude compared with that of the bending strain energy. This check showed the membrane strain energy to be negligibly small compared to the bending energy, and further verifies that the element's control of the nonlinear strains is functioning well.

The study of the stresses along the sides of the element has shown (Figure 17) that the residual membrane stresses are of forms which are necessarily produced by the BCIZ membrane functions and the lower degree nonlinear strains due to bending. The higher degree nonlinear strains due to bending have been eliminated by the HMN functions. This raises the somewhat academic question of why the BCIZ membrane functions have not completely eliminated also the lower degree nonlinear strains due to bending since it would appear that the lowest energy state would have zero values of these strains. A complete answer to this question is not known at the present time. It appears most likely that the cause lies in the use of HMN option #1; HMN option #2 would provide a more complete control of the stress state.



NODES	BOUNDARY CONDITIONS	SYMMETRY CONDITIONS
1	$\hat{u} = \hat{v} = \hat{w} = 0$ $\frac{\partial \hat{v}}{\partial \hat{x}} = \frac{\partial \hat{w}}{\partial \hat{x}} = 0$	
3	$\hat{v} = \hat{w} = 0$ $\frac{\partial \hat{v}}{\partial \hat{x}} = \frac{\partial \hat{w}}{\partial \hat{x}} = 0$	
2	$\hat{u} = 0$ $\frac{\partial \hat{w}}{\partial \hat{x}} = 0$	$\hat{v} = 0$ $\frac{\partial \hat{v}}{\partial \hat{x}} = 0$
4	$\frac{\partial \hat{w}}{\partial \hat{x}} = 0$	$\hat{v} = 0$ $\frac{\partial \hat{v}}{\partial \hat{x}} = 0$

FIGURE 13 : DEFINITION OF "TENT" PROBLEM

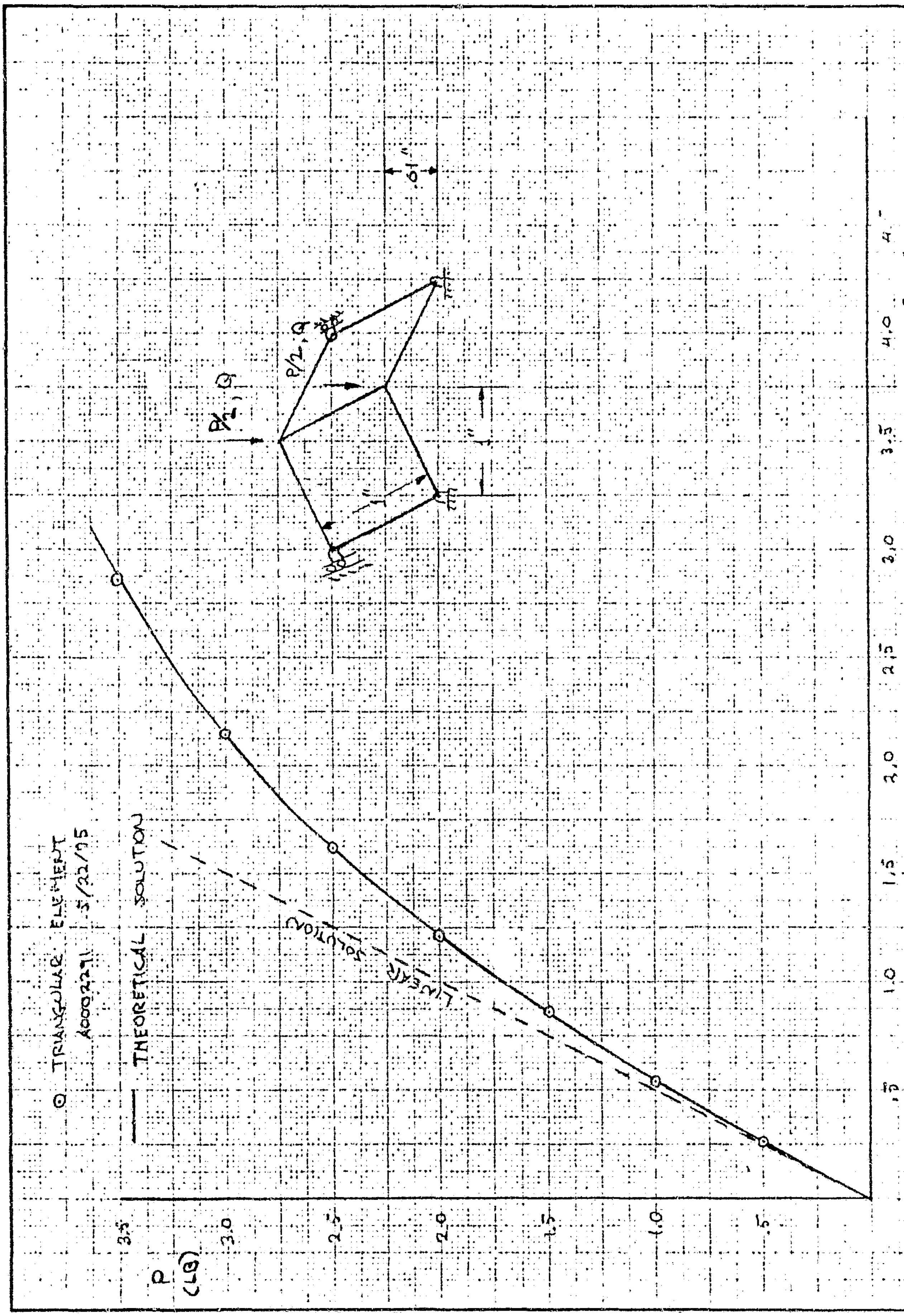
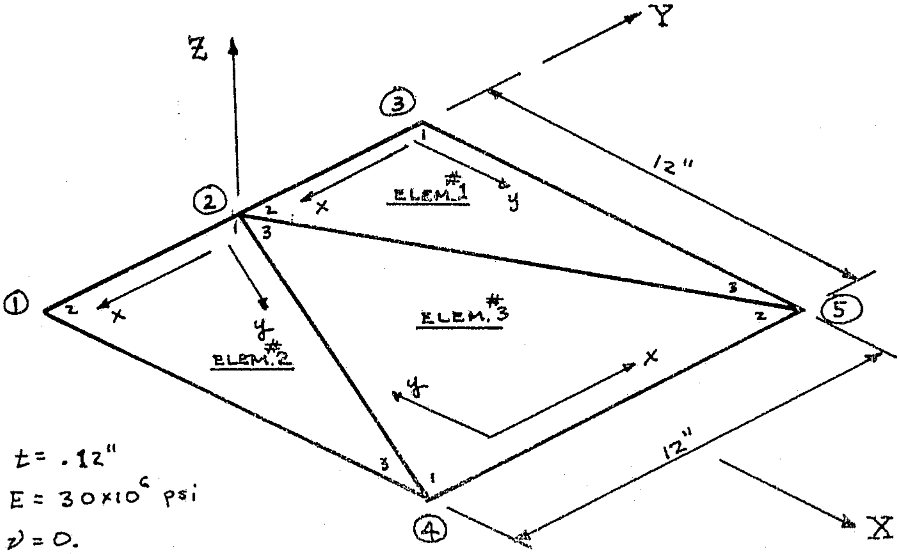


FIGURE 14: FORCE VS DEFLECTION FOR "TENT" PROBLEM

C-2



NODES	BOUNDARY CONDITIONS
1	$\hat{u} = \partial\hat{u}/\partial\hat{y} = \hat{w} = \partial\hat{w}/\partial\hat{x} = \partial\hat{w}/\partial\hat{y} = \hat{v} = \frac{\partial\hat{v}}{\partial\hat{x}} = 0$
2	$\hat{u} = \partial\hat{u}/\partial\hat{y} = \hat{v} = \partial\hat{v}/\partial\hat{x} = \hat{w} = \frac{\partial\hat{w}}{\partial\hat{x}} = \frac{\partial w}{\partial\hat{y}} = 0$
3	$\hat{u} = \partial\hat{u}/\partial\hat{y} = \hat{w} = \partial\hat{w}/\partial\hat{x} = \partial\hat{w}/\partial\hat{y} = \hat{v} = \frac{\partial\hat{v}}{\partial\hat{x}} = 0$
4	$\partial\hat{u}/\partial\hat{y} = \partial\hat{w}/\partial\hat{x} = \hat{u} = \frac{\partial\hat{v}}{\partial\hat{x}} = 0$
5	$\partial\hat{u}/\partial\hat{y} = \partial\hat{w}/\partial\hat{x} = \hat{u} = \frac{\partial\hat{v}}{\partial\hat{x}} = 0$

FIGURE 15 : DEFINITION OF BEAM PROBLEM

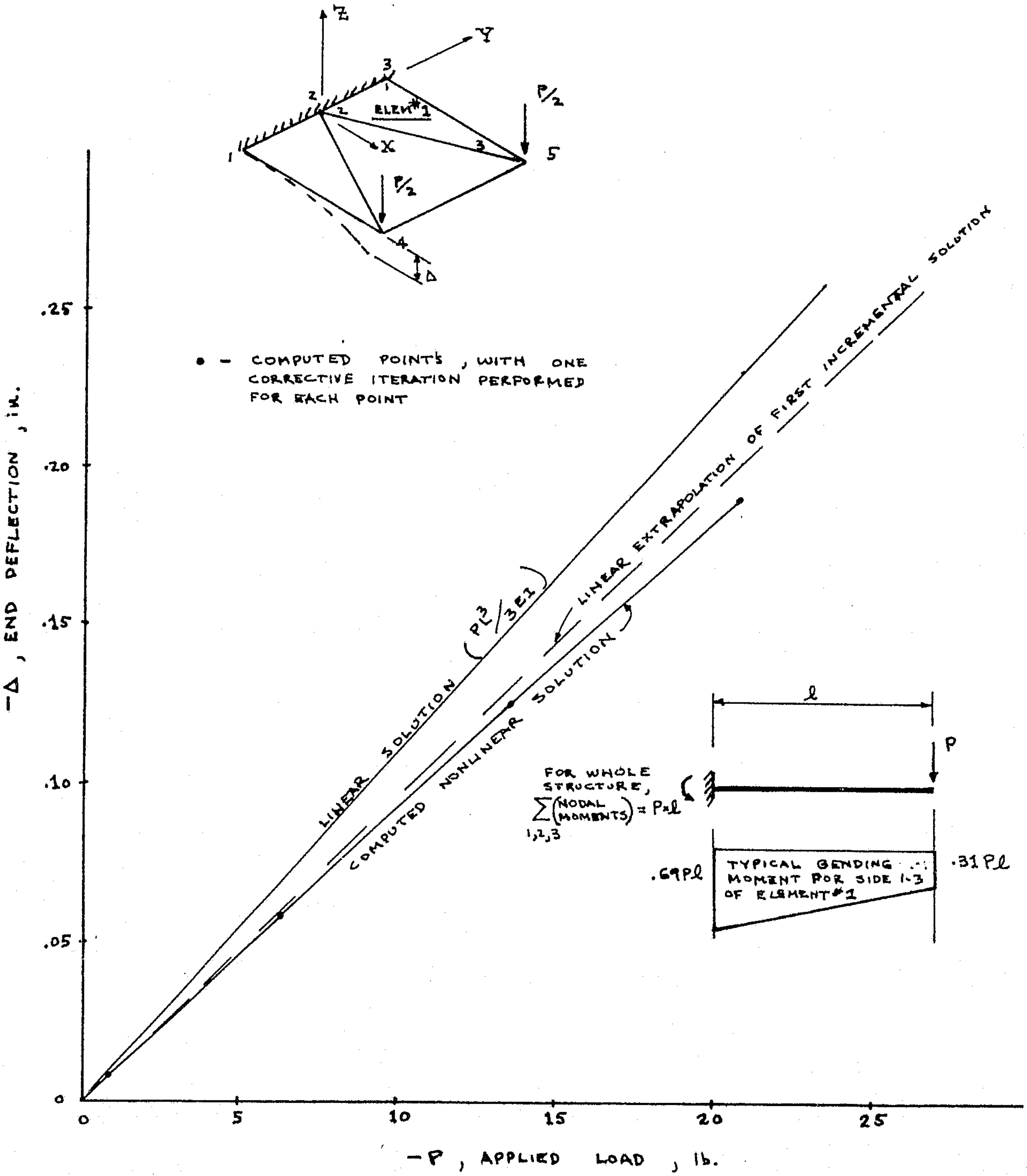


FIGURE 16 : LOAD vs DEFLECTION FOR BENDING PROBLEM

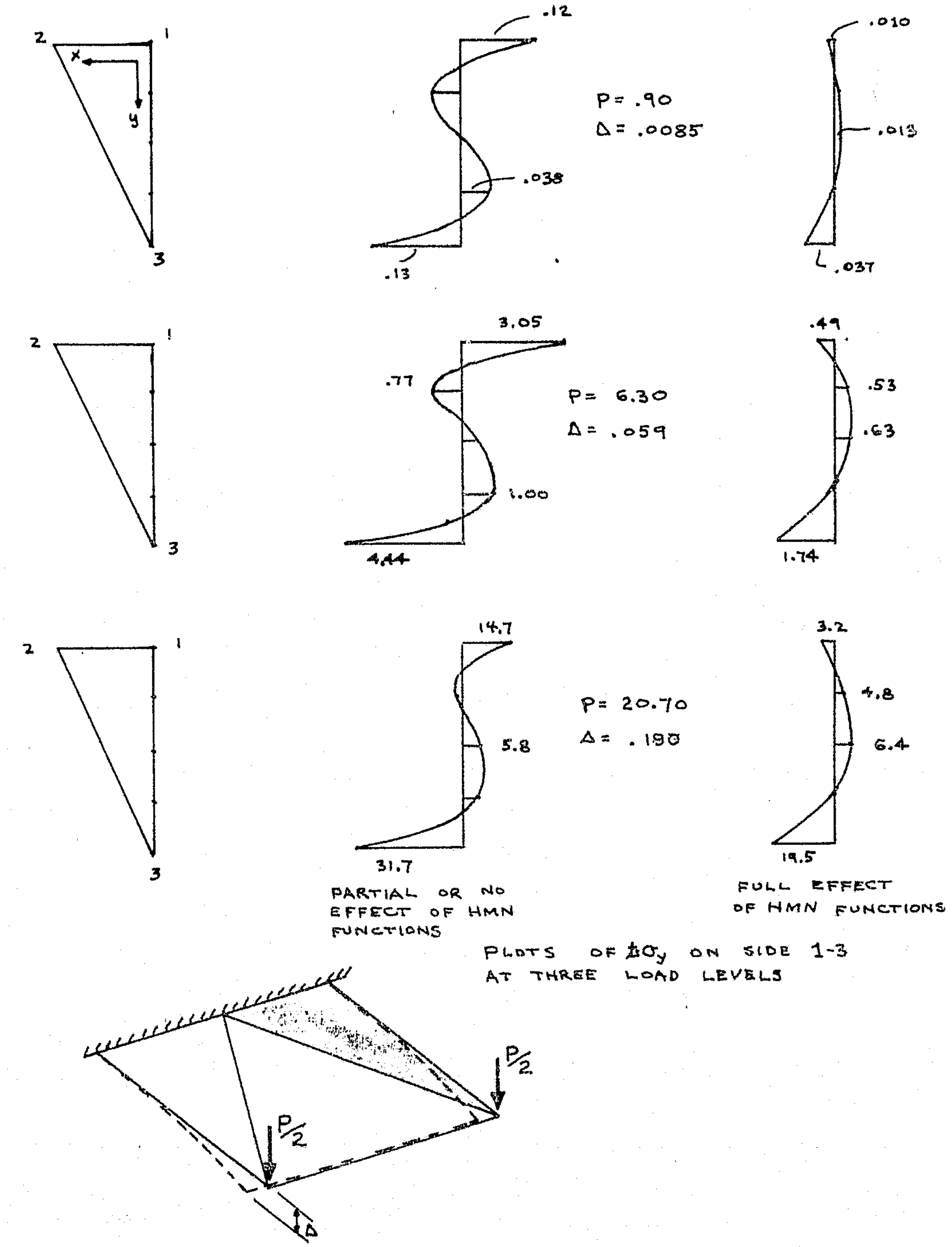


FIGURE 17 : RESIDUAL STRESSES ALONG FREE SIDE - EFFECT OF HMN FUNCTIONS

11.0 REFERENCES

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12.0 APPENDIX

12.1 TRANSFORMATION MATRICES

12.1.1 Triangular Element

<u>Global to Local Transformation</u>

Input: Nodal Coordinates

Node 1 $\left(X_1^{(0)}, Y_1^{(0)}, Z_1^{(0)}\right)$

Node 2 $\left(X_2^{(0)}, Y_2^{(0)}, Z_2^{(0)}\right)$

Node 3 $\left(X_3^{(0)}, Y_3^{(0)}, Z_3^{(0)}\right)$

Define $$X_{21}^{(0)} = X_2^{(0)} - X_1^{(0)} \tag{12.1}$$

Then the coordinates of the nodes in the local system are

$$\begin{bmatrix} x_2 & x_3 \\ 0 & y_3 \end{bmatrix} = \begin{bmatrix} \lambda_x^{(0)} & \mu_x^{(0)} & \nu_x^{(0)} \\ \lambda_y^{(0)} & \mu_y^{(0)} & \nu_y^{(0)} \end{bmatrix} \begin{bmatrix} X_{21}^{(0)} & X_{31}^{(0)} \\ Y_{21}^{(0)} & Y_{31}^{(0)} \\ Z_{21}^{(0)} & Z_{31}^{(0)} \end{bmatrix} \tag{12.2}$$

where the matrix of direction cosines are given below. For simplicity omit the superscript j denoting the step, i.e., let

$$\lambda_x = \frac{X_{21}}{\ell} \quad imply \quad \lambda_x^{(j)} = \frac{X_{21}^{(j)}}{\ell^{(j)}} \tag{12.3}$$

Then at any step j, the matrix of direction cosines is recomputed.

$$\begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{Bmatrix} = \begin{bmatrix} \lambda_x & \mu_x & \nu_x \\ \lambda_y & \mu_y & \nu_y \\ \lambda_z & \mu_z & \nu_z \end{bmatrix} \begin{Bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{Bmatrix} = [T] \begin{Bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{Bmatrix} \tag{12.4}$$

$$\begin{aligned} \lambda_x &= \frac{X_{21}}{\ell}, \quad \mu_x = \frac{Y_{21}}{\ell}, \quad \nu_x = \frac{Z_{21}}{\ell} \\ \lambda_z &= \frac{\ell_z}{\ell'}, \quad \mu_z = \frac{m_z}{\ell'}, \quad \nu_z = \frac{n_z}{\ell'} \end{aligned} \tag{12.5}$$

$$\lambda_y = \mu_z \nu_x - \mu_x \nu_z, \quad \mu_y = \nu_z \lambda_x - \nu_x \lambda_z, \quad \nu_y = \lambda_z \mu_x - \lambda_x \mu_z$$

$$\begin{aligned} \ell &= (X_{21}^2 + Y_{21}^2 + Z_{21}^2)^{1/2} \\ \ell' &= (\ell_z^2 + m_z^2 + n_z^2)^{1/2} \\ \ell_z &= (Y_{21} Z_{31} - Z_{21} Y_{31}) \\ m_z &= (Z_{21} X_{31} - X_{21} Z_{31}) \\ n_z &= (X_{21} Y_{31} - Y_{21} X_{31}) \end{aligned} \tag{12.6}$$

Global to Solution Coordinate System Transformation

At a node, average the unit normal vectors $\vec{k}$ of all M elements having this common node, Figure 2. Again it is implied that the calculation will be repeated for each solution step

$$\vec{\hat{k}} = \frac{1}{M}\sum_{M=1}^{M} \vec{k}_{(M)} = \frac{1}{M}\left(\sum_{M=1}^{M} \lfloor \lambda_{z(M)} \; \mu_{z(M)} \; \nu_{z(M)} \rfloor\right) \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} \tag{12.7}$$

$$= \lfloor \hat{\lambda}_z \; \hat{\mu}_z \; \hat{\nu}_z \rfloor \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}$$

Since $\vec{\hat{i}}$ and $\vec{\hat{j}}$ are arbitrary, they are selected to be in the XZ and YZ planes respectively.

$$\vec{\hat{i}} = \lfloor \hat{\lambda}_x \; \hat{\mu}_x \; \hat{\nu}_x \rfloor \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} \tag{12.8}$$

$$\hat{\lambda}_x = \frac{\hat{\nu}_z}{(\hat{\nu}_z^2 + \hat{\lambda}_z^2)^{1/2}}, \quad \hat{\mu}_x = 0, \quad \hat{\nu}_x = -\frac{\hat{\lambda}_z}{(\hat{\nu}_z^2 + \hat{\lambda}_z^2)^{1/2}} \tag{12.9}$$

and

$$\vec{\hat{j}} = \lfloor \hat{\lambda}_y \; \hat{\mu}_y \; \hat{\nu}_y \rfloor \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} \tag{12.10}$$

$$\hat{\lambda}_y = \hat{\mu}_z \hat{\nu}_x, \quad \hat{\mu}_y = -\hat{\lambda}_z \hat{\nu}_x + \hat{\lambda}_x \hat{\nu}_z, \quad \hat{\nu}_y = -\hat{\lambda}_x \hat{\mu}_z \tag{12.11}$$

Consequently

$$\begin{Bmatrix} \Delta\hat{u} \\ \Delta\hat{v} \\ \Delta\hat{w} \end{Bmatrix} = \begin{bmatrix} \hat{\lambda}_x & \hat{\mu}_x & \hat{\nu}_x \\ \hat{\lambda}_y & \hat{\mu}_y & \nu_y \\ \hat{\lambda}_z & \hat{\mu}_z & \nu_z \end{bmatrix} \begin{Bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{Bmatrix} = \left[\hat{T}\right] \begin{Bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{Bmatrix} \tag{12.12}$$

Solution to Local Coordinate System Transformation

From Equations (12.4) and (12.12), for each node of each element at a given load increment

$$\begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{Bmatrix} = [T][\hat{T}]^T \begin{Bmatrix} \Delta \hat{u} \\ \Delta \hat{v} \\ \Delta \hat{w} \end{Bmatrix} = [\dot{T}] \begin{Bmatrix} \Delta \hat{u} \\ \Delta \hat{v} \\ \Delta \hat{w} \end{Bmatrix} \qquad (12.13)$$

$$[\dot{T}] = \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} & \dot{T}_{13} \\ \dot{T}_{21} & \dot{T}_{22} & \dot{T}_{23} \\ \dot{T}_{31} & \dot{T}_{32} & \dot{T}_{33} \end{bmatrix} \qquad (12.14)$$

The displacement gradients transform as a second order mixed tensor

$$\frac{\partial u^i}{\partial x^j} = \frac{\partial \hat{x}^L}{\partial x^j} \frac{\partial x^i}{\partial \hat{x}^K} \frac{\partial \hat{u}^K}{\partial \hat{x}^L} \quad \text{where} \quad \begin{cases} u^1 = u, \; u^2 = v, \; u^3 = w \\ \hat{u}^1 = \hat{u}, \; \hat{u}^2 = \hat{v}, \; \hat{u}^3 = \hat{w} \\ x^1 = x, \; x^2 = y, \; x^3 = z \\ \hat{x}^1 = \hat{x}, \; \hat{x}^2 = \hat{y}, \; \hat{x}^3 = \hat{z} \end{cases} \qquad (12.15)$$

Expanding this equation and using Equation (12.13), the transformation matrix for displacement gradients for each node of each element at a given load increment is

$$\begin{Bmatrix} \theta_{uy} \\ \theta_{ux} \\ \theta_{vy} \\ \theta_{vx} \\ \theta_{wy} \\ \theta_{wx} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \\ \partial w/\partial x \\ \partial w/\partial y \end{Bmatrix} \qquad (12.16)$$

$$
\begin{Bmatrix} \partial u/\partial x \\ -\partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \\ -\partial w/\partial x \\ \partial w/\partial y \end{Bmatrix}
=
\begin{bmatrix}
\dot{T}_{11}\dot{T}_{11} & -\dot{T}_{12}\dot{T}_{11} & \dot{T}_{11}\dot{T}_{12} & \dot{T}_{12}\dot{T}_{12} & -\dot{T}_{11}\dot{T}_{13} & \dot{T}_{12}\dot{T}_{13} \\
-\dot{T}_{21}\dot{T}_{11} & \dot{T}_{22}\dot{T}_{11} & -\dot{T}_{21}\dot{T}_{12} & -\dot{T}_{22}\dot{T}_{12} & \dot{T}_{21}\dot{T}_{13} & -\dot{T}_{22}\dot{T}_{13} \\
\dot{T}_{11}\dot{T}_{21} & -\dot{T}_{12}\dot{T}_{21} & \dot{T}_{11}\dot{T}_{22} & \dot{T}_{12}\dot{T}_{22} & -\dot{T}_{11}\dot{T}_{23} & \dot{T}_{12}\dot{T}_{23} \\
\dot{T}_{21}\dot{T}_{21} & -\dot{T}_{22}\dot{T}_{21} & \dot{T}_{21}\dot{T}_{22} & \dot{T}_{22}\dot{T}_{22} & -\dot{T}_{21}\dot{T}_{23} & -\dot{T}_{22}\dot{T}_{23} \\
-\dot{T}_{11}\dot{T}_{31} & \dot{T}_{12}\dot{T}_{31} & -\dot{T}_{11}\dot{T}_{32} & -\dot{T}_{12}\dot{T}_{12} & \dot{T}_{11}\dot{T}_{33} & -\dot{T}_{12}\dot{T}_{33} \\
\dot{T}_{21}\dot{T}_{31} & -\dot{T}_{22}\dot{T}_{31} & \dot{T}_{21}\dot{T}_{32} & \dot{T}_{22}\dot{T}_{32} & -\dot{T}_{21}\dot{T}_{33} & \dot{T}_{22}\dot{T}_{33}
\end{bmatrix}
\begin{Bmatrix} \hat{\theta}_{uy} \\ \hat{\theta}_{ux} \\ \hat{\theta}_{vy} \\ \hat{\theta}_{vx} \\ \hat{\theta}_{wy} \\ \hat{\theta}_{wx} \end{Bmatrix}
\qquad (12.17)
$$

where

$$
\begin{Bmatrix} \hat{\theta}_{uy} \\ \hat{\theta}_{ux} \\ \hat{\theta}_{vy} \\ \hat{\theta}_{vx} \\ \hat{\theta}_{wy} \\ \hat{\theta}_{wx} \end{Bmatrix}
=
\begin{Bmatrix} \partial \hat{u}/\partial \hat{x} \\ -\partial \hat{u}/\partial \hat{y} \\ \partial \hat{v}/\partial \hat{x} \\ \partial \hat{v}/\partial \hat{y} \\ -\partial \hat{w}/\partial \hat{x} \\ \partial \hat{w}/\partial \hat{y} \end{Bmatrix}
\qquad (12.18)
$$

The derivatives with respect to $\hat{X}_3$ have been deleted from the stiffness matrix. The matrices given in Equations (12.12) and (12.18) can then be re-ordered and assembled for each node to give the transformation matrix

12.1.2 Quadrilateral Element

Mean Plane Calculation and Global to Local Transformation

Input: Nodal Coordinates

Node 1 $\left(X_1^{(0)}, Y_1^{(0)}, Z_1^{(0)}\right)$

Node 2 $\left(X_2^{(0)}, Y_2^{(0)}, Z_2^{(0)}\right)$

⋮

Node 8 $\left(X_8^{(0)}, Y_8^{(0)}, Z_8^{(0)}\right)$

A mean plane can be shown to exist that passes through the midpoints of the straight lines joining the corner nodes, Figure 1 . For simplicity, the superscript denoting the step is omitted since the mean plane and the matrix of direction cosines is recomputed at each step.

$$\begin{aligned}
X_a &= \tfrac{1}{2}(X_1 + X_7) & Y_a &= \tfrac{1}{2}(Y_1 + Y_7) & Z_a &= \tfrac{1}{2}(Z_1 + Z_7) \\
X_b &= \tfrac{1}{2}(X_3 + X_5) & Y_b &= \tfrac{1}{2}(Y_3 + Y_5) & Z_b &= \tfrac{1}{2}(Z_3 + Z_5) \\
X_c &= \tfrac{1}{2}(X_1 + X_3) & Y_c &= \tfrac{1}{2}(Y_1 + Y_3) & Z_c &= \tfrac{1}{2}(Z_1 + Z_3) \\
X_d &= \tfrac{1}{2}(X_5 + X_7) & Y_d &= \tfrac{1}{2}(Y_5 + Y_7) & Z_d &= \tfrac{1}{2}(Z_5 + Z_7)
\end{aligned} \tag{12.19}$$

The matrix of direction cosines is

$$\begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{Bmatrix} = \begin{bmatrix} \lambda_x & \mu_x & \nu_x \\ \lambda_y & \mu_y & \nu_y \\ \lambda_z & \mu_z & \nu_z \end{bmatrix} \begin{Bmatrix} \Delta U \\ \Delta V \\ \Delta Z \end{Bmatrix} \tag{12.20}$$

where defining

$$X_{ba} = X_b - X_a \tag{12.21}$$

$$\lambda_x = \frac{X_{ba}}{\ell} \qquad \mu_x = \frac{Y_{ba}}{\ell} \qquad \nu_x = \frac{Z_{ba}}{\ell}$$

$$\lambda_z = \frac{\ell_{ab}}{D_{ab}} \qquad \mu_z = \frac{M_{ab}}{D_{ab}} \qquad \nu_z = \frac{n_{ab}}{D_{ab}} \tag{12.22}$$

$$\lambda_y = \mu_z \nu_x - \mu_x \nu_z , \quad \mu_y = \nu_z \lambda_x - \nu_x \lambda_z , \quad \nu_y = \lambda_z \mu_x - \lambda_x \mu_z \tag{12.22 (CONT.)}$$

$$\ell = (X_{ba}^2 + Y_{ba}^2 + Z_{ba}^2)^{1/2}$$

$$\ell_{ab} = Y_{ba} Z_{dc} - Y_{dc} Z_{ba}, \quad M_{ab} = X_{dc} Z_{ba} - X_{ba} Z_{dc} , \quad n_{ab} = X_{ab} Y_{dc} - X_{dc} Y_{ba} \tag{12.23}$$

$$D_{ab} = (\ell_{ab}^2 + M_{ab}^2 - n_{ab}^2)^{1/2} \tag{12.24}$$

The coordinates of the nodes in the local coordinate system can then be calculated using the matrix of direction cosines and the defined origin of the local coordinates

$$X_o = \tfrac{1}{2}(X_a + X_b), \; Y_o = \tfrac{1}{2}(Y_a + Y_b), \; Z_o = \tfrac{1}{2}(Z_a + Z_b) \tag{12.25}$$

$$\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} = [T] \begin{Bmatrix} X_i - X_o \\ Y_i - Y_o \\ Z_i - Z_o \end{Bmatrix} \tag{12.26}$$

where the Z coordinate is the initial shape w^o.

Solution to Local Coordinate System Transformation

The translational degrees of freedom transform like the triangular element, Equation 12.13. The rotational degrees of freedom transform as vectors, so that at each node of each element of a given load increment the transformation has the form

$$\begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta \theta_x \\ \Delta \theta_y \end{Bmatrix} = \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} & \dot{T}_{13} & 0 & 0 \\ \dot{T}_{21} & \dot{T}_{22} & \dot{T}_{23} & 0 & 0 \\ \dot{T}_{31} & \dot{T}_{32} & \dot{T}_{33} & 0 & 0 \\ 0 & 0 & 0 & \dot{T}_{11} & \dot{T}_{12} \\ 0 & 0 & 0 & \dot{T}_{21} & \dot{T}_{22} \end{bmatrix} \begin{Bmatrix} \Delta \hat{u} \\ \Delta \hat{v} \\ \Delta \hat{w} \\ \Delta \hat{\theta}_x \\ \Delta \hat{\theta}_y \end{Bmatrix} \tag{12.27}$$

where rotations about the $\hat{Z}$ axis ($\Delta \hat{\theta}_z$) have been deleted from the stiffness matrix.

12.2 TRIANGULAR ELEMENT GO MATRIX

The area coordinates are related to rectangular cartesian coordinates (Figure 9) by

$$\begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{Bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad (12.28)$$

$$\begin{aligned} e_{11} &= x_2 y_3 / 2A \\ e_{12} &= -y_3 / 2A \\ e_{13} &= (x_3 - x_2)/2A \\ e_{22} &= y_3 / 2A \\ e_{23} &= -x_3 / 2A \\ e_{33} &= x_2 / 2A \\ e_{21} &= e_{31} = e_{32} = 0 \end{aligned} \qquad (12.29)$$

Then the GO matrix is given by

$$\begin{aligned} GO_{5,7} &= e_{12} \\ GO_{2,2} &= e_{22} \\ GO_{5,25} &= e_{32} = 0 \\ GO_{5,8} &= \frac{\partial F_1}{\partial x} \\ GO_{5,9} &= \frac{\partial F_2}{\partial x} \\ GO_{5,17} &= \frac{\partial F_3}{\partial x} \\ GO_{5,18} &= \frac{\partial F_4}{\partial x} \end{aligned} \qquad (12.30)$$

$$GO_{5,26} = \frac{\partial F_5}{\partial x}$$

$$GO_{5,27} = \frac{\partial F_6}{\partial x}$$

$$GO_{6,7} = e_{13}$$

$$GO_{6,16} = e_{23}$$

$$GO_{6,25} = e_{33}$$

$$GO_{6,8} = \frac{\partial F_1}{\partial y}$$

$$GO_{6,9} = \frac{\partial F_2}{\partial y}$$

$$GO_{6,17} = \frac{\partial F_3}{\partial y}$$

$$GO_{6,18} = \frac{\partial F_4}{\partial y}$$

$$GO_{6,26} = \frac{\partial F_5}{\partial y}$$

$$GO_{6,27} = \frac{\partial F_6}{\partial y}$$

$$GO_{7,8} = \frac{\partial^2 F_1}{\partial x^2}$$

$$GO_{7,9} = \frac{\partial^2 F_2}{\partial x^2}$$

$$GO_{7,17} = \frac{\partial^2 F_3}{\partial x^2}$$

$$GO_{7,18} = \frac{\partial^2 F_4}{\partial x^2}$$

$$GO_{7,26} = \frac{\partial^2 F_5}{\partial x^2}$$

$$GO_{7,27} = \frac{\partial^2 F_6}{\partial x^2}$$

$$GO_{8,8} = \frac{\partial^2 F_1}{\partial y^2}$$
$$GO_{8,9} = \frac{\partial^2 F_2}{\partial y^2}$$
$$GO_{8,17} = \frac{\partial^2 F_3}{\partial y^2}$$
$$GO_{8,18} = \frac{\partial^2 F_4}{\partial y^2}$$
$$GO_{8,26} = \frac{\partial^2 F_5}{\partial y^2}$$
$$GO_{8,27} = \frac{\partial^2 F_6}{\partial y^2}$$

$$GO_{9,8} = \frac{\partial^2 F_1}{\partial x \partial y}$$
$$GO_{9,9} = \frac{\partial^2 F_2}{\partial x \partial y}$$
$$GO_{9,17} = \frac{\partial^2 F_3}{\partial x \partial y}$$
$$GO_{9,18} = \frac{\partial^2 F_4}{\partial x \partial y}$$
$$GO_{9,26} = \frac{\partial^2 F_5}{\partial x \partial y}$$
$$GO_{9,27} = \frac{\partial^2 F_6}{\partial x \partial y}$$

$$GO_{1,1} = GO_{5,7}$$
$$GO_{1,2} = GO_{5,8}$$
$$GO_{1,12} = GO_{5,18}$$
$$GO_{1,40} = \frac{\partial G_1}{\partial x}$$
$$GO_{1,28} = \frac{\partial G_4}{\partial x}$$
$$GO_{1,46} = \cos\alpha_1 \frac{\partial H_{11}}{\partial x}$$
$$GO_{1,47} = \cos\alpha_2 \frac{\partial H_{22}}{\partial x}$$

$$GO_{1,10} = GO_{5,16}$$
$$GO_{1,3} = GO_{5,9}$$
$$GO_{1,20} = GO_{5,26}$$
$$GO_{1,42} = \frac{\partial G_2}{\partial x}$$
$$GO_{1,30} = \frac{\partial G_5}{\partial x}$$
$$GO_{1,47} = \cos\alpha_1 \frac{\partial H_{21}}{\partial x}$$
$$GO_{1,50} = \cos\alpha_3 \frac{\partial H_{13}}{\partial x}$$

$$GO_{1,19} = GO_{5,25}$$
$$GO_{1,11} = GO_{5,17}$$
$$GO_{1,21} = GO_{5,27}$$
$$GO_{1,44} = \frac{\partial G_3}{\partial x}$$
$$GO_{1,32} = \frac{\partial G_6}{\partial x}$$
$$GO_{1,48} = \cos\alpha_2 \frac{\partial H_{12}}{\partial x}$$
$$GO_{1,51} = \cos\alpha_3 \frac{\partial H_{23}}{\partial x}$$

$$GO_{1,34} = -\sin\alpha_1 \frac{\partial H_{11}}{\partial x} \qquad GO_{1,35} = -\sin\alpha_1 \frac{\partial H_{21}}{\partial x} \qquad GO_{1,36} = -\sin\alpha_2 \frac{\partial H_{12}}{\partial x}$$

$$GO_{1,37} = -\sin\alpha_2 \frac{\partial H_{22}}{\partial x} \qquad GO_{1,38} = -\sin\alpha_3 \frac{\partial H_{13}}{\partial x} \qquad GO_{1,39} = -\sin\alpha_3 \frac{\partial H_{23}}{\partial x}$$

$$GO_{2,1} = GO_{6,7} \qquad GO_{2,10} = GO_{6,16} \qquad GO_{2,19} = GO_{6,25}$$

$$GO_{22} = GO_{6,8} \qquad GO_{2,3} = GO_{6,9} \qquad GO_{2,11} = GO_{6,17}$$

$$GO_{2,12} = GO_{6,18} \qquad GO_{2,20} = GO_{6,26} \qquad GO_{2,21} = GO_{6,27}$$

$$GO_{2,40} = \frac{\partial G_1}{\partial y} \qquad GO_{2,42} = \frac{\partial G_2}{\partial y} \qquad GO_{2,44} = \frac{\partial G_3}{\partial y}$$

$$GO_{2,28} = \frac{\partial G_4}{\partial y} \qquad GO_{2,30} = \frac{\partial G_5}{\partial y} \qquad GO_{2,32} = \frac{\partial G_6}{\partial y}$$

$$GO_{2,46} = \cos\alpha_1 \frac{\partial H_{11}}{\partial y} \qquad GO_{2,47} = \cos\alpha_1 \frac{\partial H_{21}}{\partial y} \qquad GO_{2,48} = \cos\alpha_2 \frac{\partial H_{12}}{\partial y}$$

$$GO_{2,49} = \cos\alpha_2 \frac{\partial H_{22}}{\partial y} \qquad GO_{2,50} = \cos\alpha_3 \frac{\partial H_{13}}{\partial y} \qquad GO_{2,51} = \cos\alpha_3 \frac{\partial H_{23}}{\partial y}$$

$$GO_{2,34} = -\sin\alpha_1 \frac{\partial H_{11}}{\partial y} \qquad GO_{2,35} = -\sin\alpha_1 \frac{\partial H_{21}}{\partial y} \qquad GO_{2,36} = -\sin\alpha_2 \frac{\partial H_{12}}{\partial y}$$

$$GO_{2,32} = -\sin\alpha_2 \frac{\partial H_{22}}{\partial y} \qquad GO_{2,38} = -\sin\alpha_3 \frac{\partial H_{13}}{\partial y} \qquad GO_{2,39} = -\sin\alpha_3 \frac{\partial H_{23}}{\partial y}$$

$$GO_{3,4} = GO_{5,7} \qquad GO_{3,13} = GO_{5,16} \qquad GO_{3,22} = GO_{5,25}$$

$$GO_{3,5} = GO_{5,8} \qquad GO_{3,6} = GO_{5,9} \qquad GO_{3,14} = GO_{5,17}$$

$$GO_{3,15} = GO_{5,18} \qquad GO_{3,23} = GO_{5,26} \qquad GO_{3,24} = GO_{5,27}$$

$$GO_{3,41} = GO_{1,40} \qquad GO_{3,43} = GO_{1,42} \qquad GO_{3,45} = GO_{1,44}$$

$$GO_{3,29} = GO_{1,28} \qquad GO_{3,31} = GO_{1,30} \qquad GO_{3,33} = GO_{1,32}$$

$$GO_{3,46} = GO_{1,34} \qquad GO_{3,47} = -GO_{1,35} \qquad GO_{3,48} = -GO_{1,36}$$

$$GO_{3,49} = GO_{1,37} \qquad GO_{3,50} = -GO_{1,38} \qquad GO_{3,51} = -GO_{1,39}$$

$GO_{1,46} = GO_{1,46}$ $\qquad GO_{3,35} = GO_{1,47}$ $\qquad GO_{3,36} = GO_{1,48}$

$GO_{3,34} = GO_{1,49}$ $\qquad GO_{3,38} = GO_{1,50}$ $\qquad GO_{3,39} = GO_{1,51}$

$GO_{4,4} = GO_{6,7}$ $\qquad GO_{4,13} = GO_{6,16}$ $\qquad GO_{4,22} = GO_{6,25}$

$GO_{4,5} = GO_{6,8}$ $\qquad GO_{4,6} = GO_{6,9}$ $\qquad GO_{4,14} = GO_{6,17}$

$GO_{4,15} = GO_{6,18}$ $\qquad GO_{4,23} = GO_{6,26}$ $\qquad GO_{4,24} = GO_{6,27}$

$GO_{4,41} = GO_{2,40}$ $\qquad GO_{4,43} = GO_{2,42}$ $\qquad GO_{4,45} = GO_{2,44}$

$GO_{4,29} = GO_{2,28}$ $\qquad GO_{4,31} = GO_{2,30}$ $\qquad GO_{4,33} = GO_{2,32}$

$GO_{4,46} = -GO_{2,34}$ $\qquad GO_{4,47} = -GO_{2,35}$ $\qquad GO_{4,48} = -GO_{2,36}$

$GO_{4,49} = -GO_{2,37}$ $\qquad GO_{4,50} = -GO_{2,38}$ $\qquad GO_{4,51} = -GO_{2,39}$

$GO_{4,34} = GO_{2,46}$ $\qquad GO_{4,35} = GO_{2,47}$ $\qquad GO_{4,36} = GO_{2,48}$

$GO_{4,37} = GO_{2,49}$ $\qquad GO_{4,38} = GO_{2,50}$ $\qquad GO_{4,39} = GO_{2,51}$

12.3 TRIANGULAR ELEMENT A0 & A1 MATRICES

$$
\begin{array}{lll}
A0_{11} = 1 & A1_{155} = 1 & \\
A0_{15} = \dfrac{\partial w^{\circ}}{\partial x} & & \\
A0_{24} = 1 & A1_{266} = 1 & \\
A0_{26} = \dfrac{\partial w^{\circ}}{\partial x} & & \\
A0_{32} = 1 & A1_{356} = 1 & A1_{365} = 1 \\
A0_{33} = 1 & & \\
A0_{35} = \dfrac{\partial w^{\circ}}{\partial y} & & \\
A0_{36} = \dfrac{\partial w^{\circ}}{\partial x} & & \\
A0_{41} = -\dfrac{\partial^2 w^{\circ}}{\partial x^2} & A1_{417} = -1 & A1_{471} = -1 \\
A0_{43} = -\dfrac{\partial^2 w^{\circ}}{\partial x\, \partial y} & A1_{439} = -1 & A1_{493} = -1 \\
A0_{47} = -1 & & \\
A0_{52} = -\dfrac{\partial^2 w^{\circ}}{\partial x\, \partial y} & A1_{529} = -1 & A1_{592} = -1 \\
A0_{54} = -\dfrac{\partial^2 w^{\circ}}{\partial y^2} & A1_{548} = -1 & A1_{548} = -1 \\
A0_{58} = -1 & & \\
A0_{61} = -\dfrac{\partial^2 w^{\circ}}{\partial x\, \partial y} & A1_{619} = -1 & A1_{691} = -1 \\
A0_{62} = -\dfrac{\partial^2 w^{\circ}}{\partial x^2} & A1_{627} = -1 & A1_{672} = -1 \\
A0_{63} = -\dfrac{\partial^2 w^{\circ}}{\partial y^2} & A1_{638} = -1 & A1_{683} = -1 \\
A0_{64} = -\dfrac{\partial^2 w^{\circ}}{\partial x\, \partial y} & A1_{649} = -1 & A1_{694} = -1 \\
A0_{69} = -1 & &
\end{array}
\qquad (12.31)
$$

12.4 TRIANGULAR ELEMENT D0 MATRIX

$$[D0] = \begin{bmatrix} \frac{Et}{1-\nu^2} & \frac{E\nu t}{1-\nu^2} & 0 & 0 & 0 & 0 \\ \frac{E\nu t}{1-\nu^2} & \frac{Et}{1-\nu^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{Et}{2(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{Et^3}{12(1-\nu^2)} & \frac{E\nu t^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{E\nu t}{12(1-\nu^2)} & \frac{Et^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Et^3}{24(1+\nu)} \end{bmatrix} \qquad (12.32)$$

12.5 TRIANGULAR ELEMENT HMN EQUATIONS

Depending on the option exercised, the HMN equations have the form

$$[H]\begin{Bmatrix}\Delta\alpha_3\\ \Delta\alpha_2\end{Bmatrix} = [HO]\begin{Bmatrix}\Delta\theta_\omega^*\\ \Delta\alpha_1\end{Bmatrix} + [H1(\theta_\omega^*)]\{\Delta\theta_\omega^*\} \qquad (12.33)$$

0x0	0x0	0x0
or 6x6	or 6x12	or 6x6
or 18x18	or 18x12	or 18x6

where [H1] is the product of a triply subscripted matrix and the column of total deformational rotations accumulated in the previous n-1 steps.

$$H1\left(\theta_\omega^*\right)_{ij} = \overline{H1}_{ijk}\ \theta_{\omega_k}^{*(n-1)} \qquad (12.34)$$

The [H] matrix must then be inverted, expanded and reordered to form the constraint equation.

The [H], [HO] and $\overline{H1}_{ijk}$ matrices are presented on the following pages. Terms that are not written explicitly are zero.

$$
[H] = \begin{array}{c}
\begin{array}{cccccccccccccccccc}
\Delta C_{11} & \Delta C_{21} & \Delta C_{12} & \Delta C_{22} & \Delta C_{13} & \Delta C_{23} & \Delta d_{11} & \Delta d_{21} & \Delta d_{12} & \Delta d_{22} & \Delta d_{13} & \Delta d_{23} & a_1 & b_1 & a_2 & b_2 & a_3 & b_3
\end{array} \\
\left[\begin{array}{cccccccccccccccccc}
2(N_3^{(1)}-N_2^{(1)}) & 12N_2^{(1)}-8N_3^{(1)} & & & & & & & & & & & & & & & & \\
& 5(N_3^{(1)}-N_2^{(1)}) & & & & & & & & & & & & & & & & \\
& & 2(N_1^{(2)}-N_3^{(2)}) & 12N_3^{(2)}-8N_1^{(2)} & & & & & & & & & & & & & & \\
& & & 5(N_1^{(2)}-N_3^{(2)}) & & & & & & & & & & & & & & \\
& & & & 2(N_2^{(3)}-N_1^{(3)}) & 12N_1^{(3)}-8N_2^{(3)} & & & & & & & & & & & & \\
& & & & & 5(N_2^{(3)}-N_1^{(3)}) & & & & & & & & & & & & \\
& & & & & & 2(S_3^{(1)}-S_2^{(1)}) & 12S_2^{(1)}-8S_3^{(1)} & & & & & & & -2S_1^{(1)}\sin\alpha_1 & 2S_1^{(1)}\cos\alpha_1 & & \\
& & & & & & & 5(S_3^{(1)}-S_2^{(1)}) & & & & & & & S_1^{(1)}\sin\alpha_1 & -S_1^{(1)}\cos\alpha_1 & & \\
& & & & & & & & 2(S_1^{(2)}-S_3^{(2)}) & 12S_3^{(2)}-8S_1^{(2)} & & & & & & & -2S_2^{(2)}\sin\alpha_2 & 2S_2^{(2)}\cos\alpha_2 \\
& & & & & & & & & 5(S_1^{(2)}-S_3^{(2)}) & & & & & & & S_2^{(2)}\sin\alpha_2 & -S_2^{(2)}\cos\alpha_2 \\
& & & & & & & & & & 2(S_2^{(3)}-S_1^{(3)}) & 12S_1^{(3)}-8S_2^{(3)} & -2S_3^{(3)}\sin\alpha_3 & 2S_3^{(3)}\cos\alpha_3 & & & & \\
& & & & & & & & & & & 5(S_2^{(3)}-S_1^{(3)}) & S_3^{(3)}\sin\alpha_3 & -S_3^{(3)}\cos\alpha_3 & & & & \\
2(S_3^{(1)}-S_2^{(1)}) & 12S_2^{(1)}-8S_3^{(1)} & & & & & 2(N_3^{(1)}-N_2^{(1)}) & 12N_2^{(1)}-8N_3^{(1)} & & & & & & & 2(S_1^{(1)}\cos\alpha_1-N_1^{(1)}\sin\alpha_1) & 2(S_1^{(1)}\sin\alpha_1+N_1^{(1)}\cos\alpha_1) & & \\
& 5(S_3^{(1)}-S_2^{(1)}) & & & & & & 5(N_3^{(1)}-N_2^{(1)}) & & & & & & & -S_1^{(1)}\cos\alpha_1+N_1^{(1)}\sin\alpha_1 & -S_1^{(1)}\sin\alpha_1-N_1^{(1)}\cos\alpha_1 & & \\
& & 2(S_1^{(2)}-S_3^{(2)}) & 12S_3^{(2)}-8S_1^{(2)} & & & & & 2(N_1^{(2)}-N_3^{(2)}) & 12N_3^{(2)}-8N_1^{(2)} & & & & & & & 2(S_2^{(2)}\cos\alpha_2-N_2^{(2)}\sin\alpha_2) & 2(S_2^{(2)}\sin\alpha_2+N_2^{(2)}\cos\alpha_2) \\
& & & 5(S_1^{(2)}-S_3^{(2)}) & & & & & & 5(N_1^{(2)}-N_3^{(2)}) & & & & & & & -S_2^{(2)}\cos\alpha_2+N_2^{(2)}\sin\alpha_2 & -S_2^{(2)}\sin\alpha_2-N_2^{(2)}\cos\alpha_2 \\
& & & & 2(S_2^{(3)}-S_1^{(3)}) & 12S_1^{(3)}-8S_2^{(3)} & & & & & 2(N_2^{(3)}-N_1^{(3)}) & 12N_1^{(3)}-8N_2^{(3)} & 2(S_3^{(3)}\cos\alpha_3-N_3^{(3)}\sin\alpha_3) & 2(S_3^{(3)}\sin\alpha_3+N_3^{(3)}\cos\alpha_3) & & & & \\
& & & & & 5(S_2^{(3)}-S_1^{(3)}) & & & & & & 5(N_2^{(3)}-N_1^{(3)}) & -S_3^{(3)}\cos\alpha_3+N_3^{(3)}\sin\alpha_3 & -S_3^{(3)}\sin\alpha_3-N_3^{(3)}\cos\alpha_3 & & & &
\end{array}\right]
\end{array}
\qquad (12.35)
$$

$$
[HO] = \begin{array}{c}
\begin{array}{cccccccccccc}
\Delta\theta^*_{\omega_1} & \Delta\theta^*_{\omega_2} & \Delta\theta^*_{\omega_3} & \Delta\theta^*_{\omega_4} & \Delta\theta^*_{\omega_5} & \Delta\theta^*_{\omega_6} & a_4 & b_4 & a_5 & b_5 & a_6 & b_6
\end{array} \\
\left[\begin{array}{cccccccccccc}
\dot{A}_1\dot{F}_{11z} & \dot{A}_1\dot{F}_{21z} & \dot{A}_1\dot{F}_{31z} & \dot{A}_1\dot{F}_{41z} & \dot{A}_1\dot{F}_{51z} & \dot{A}_1\dot{F}_{61z} & & & & & & \\
\dot{A}_2\dot{F}_{12z} & \dot{A}_2\dot{F}_{22z} & \dot{A}_2\dot{F}_{32z} & \dot{A}_2\dot{F}_{42z} & \dot{A}_2\dot{F}_{52z} & \dot{A}_2\dot{F}_{62z} & & & & & & \\
\dot{A}_3\dot{F}_{13z} & \dot{A}_3\dot{F}_{23z} & \dot{A}_3\dot{F}_{33z} & \dot{A}_3\dot{F}_{43z} & \dot{A}_3\dot{F}_{53z} & \dot{A}_3\ddot{F}_{63z} & & & & & & \\
\bar{A}_1\bar{F}_{11z} & \bar{A}_1\bar{F}_{21z} & \bar{A}_1\bar{F}_{31z} & \bar{A}_1\bar{F}_{41z} & \bar{A}_1\bar{F}_{51z} & \bar{A}_1\bar{F}_{61z} & & & -S_1^{(1)}\sin\alpha_1 & +S_1^{(1)}\cos\alpha_1 & -3S_1^{(1)}\sin\alpha_1 & 3S_1^{(1)}\cos\alpha_1 \\
 & & & & & & & & S_1^{(1)}\sin\alpha_1 & -S_1^{(1)}\cos\alpha_1 & S_1^{(1)}\sin\alpha_1 & -S_1^{(1)}\cos\alpha_1 \\
\bar{A}_2\bar{F}_{12z} & \bar{A}_2\bar{F}_{22z} & \bar{A}_2\bar{F}_{32z} & \bar{A}_2\bar{F}_{42z} & \bar{A}_2\bar{F}_{52z} & \bar{A}_2\bar{F}_{62z} & -3S_2^{(2)}\sin\alpha_2 & 3S_2^{(2)}\cos\alpha_2 & & & -S_2^{(2)}\sin\alpha_2 & S_2^{(2)}\cos\alpha_2 \\
 & & & & & & S_2^{(2)}\sin\alpha_2 & -S_2^{(2)}\cos\alpha_2 & & & S_2^{(2)}\sin\alpha_2 & -S_2^{(2)}\cos\alpha_2 \\
\bar{A}_3\bar{F}_{13z} & \bar{A}_3\bar{F}_{23z} & \bar{A}_3\bar{F}_{33z} & \bar{A}_3\bar{F}_{43z} & \bar{A}_3\bar{F}_{53z} & \bar{A}_3\bar{F}_{63z} & -S_3^{(3)}\sin\alpha_3 & S_3^{(3)}\cos\alpha_3 & -3S_3^{(3)}\sin\alpha_3 & 3S_3^{(3)}\cos\alpha_3 & & \\
 & & & & & & S_3^{(3)}\sin\alpha_3 & -S_3^{(3)}\cos\alpha_3 & S_3^{(3)}\sin\alpha_3 & -S_3^{(3)}\cos\alpha_3 & & \\
\dot{A}_1\bar{F}_{11z}+ & \dot{A}_1\bar{F}_{21z}+ & \dot{A}_1\bar{F}_{31z}+ & \dot{A}_1\bar{F}_{41z}+ & \dot{A}_1\bar{F}_{51z}+ & \dot{A}_1\bar{F}_{61z}+ & & & S_1^{(1)}\cos\alpha_1- & S_1^{(1)}\sin\alpha_1+ & 3(S_1^{(1)}\cos\alpha_1- & 3(S_1^{(1)}\sin\alpha_1+ \\
+\bar{A}_1\dot{F}_{11z} & +\bar{A}_1\dot{F}_{21z} & +\bar{A}_1\dot{F}_{31z} & +\bar{A}_1\dot{F}_{41z} & +\bar{A}_1\dot{F}_{51z} & +\bar{A}_1\dot{F}_{61z} & & & N_1^{(1)}\sin\alpha_1 & +N_1^{(1)}\cos\alpha_1 & N_1^{(1)}\sin\alpha_1) & +N_1^{(1)}\cos\alpha_1) \\
 & & & & & & & & -S_1^{(1)}\cos\alpha_1+ & -S_1^{(1)}\sin\alpha_1- & -S_1^{(1)}\cos\alpha_1+ & -S_1^{(1)}\sin\alpha_1- \\
 & & & & & & & & +N_1^{(1)}\sin\alpha_1 & N_1^{(1)}\cos\alpha_1 & +N_1^{(1)}\cos\alpha_1 & N_1^{(1)}\cos\alpha_1 \\
\dot{A}_2\bar{F}_{12z}+ & \dot{A}_2\bar{F}_{22z}+ & \dot{A}_2\bar{F}_{32z}+ & \dot{A}_2\bar{F}_{42z}+ & \dot{A}_2\bar{F}_{52z}+ & \dot{A}_2\bar{F}_{62z}+ & 3(S_2^{(2)}\cos\alpha_2- & 3(S_2^{(2)}\sin\alpha_2+ & & & S_2^{(2)}\cos\alpha_2- & S_2^{(2)}\sin\alpha_2+ \\
+\bar{A}_2\dot{F}_{12z} & +\bar{A}_2\dot{F}_{22z} & +\bar{A}_2\dot{F}_{32z} & +\bar{A}_2\dot{F}_{42z} & +\bar{A}_2\dot{F}_{52z} & +\bar{A}_2\dot{F}_{62z} & N_2^{(2)}\sin\alpha_2) & N_2^{(2)}\cos\alpha_2) & & & N_2^{(2)}\sin\alpha_2 & +N_2\cos\alpha_2 \\
 & & & & & & -S_2^{(2)}\cos\alpha_2+ & -S_2^{(2)}\sin\alpha_2- & & & -S_2^{(2)}\cos\alpha_2+ & -S_2^{(2)}\sin\alpha_2- \\
 & & & & & & +N_2^{(2)}\sin\alpha_2 & N_2^{(2)}\cos\alpha_2 & & & +N_2^{(2)}\sin\alpha_2 & N_2^{(2)}\cos\alpha_2 \\
\dot{A}_3\bar{F}_{13z}+ & \dot{A}_3\bar{F}_{23z}+ & \dot{A}_3\bar{F}_{33z}+ & \dot{A}_3\bar{F}_{43z}+ & \dot{A}_3\bar{F}_{53z}+ & \dot{A}_3\bar{F}_{63z}+ & S_3^{(3)}\cos\alpha_3- & S_3^{(3)}\sin\alpha_3+ & 3(S_3^{(3)}\cos\alpha_3- & 3(S_3^{(3)}\sin\alpha_3+ & & \\
+\bar{A}_3\dot{F}_{13z} & +\bar{A}_3\dot{F}_{23z} & +\bar{A}_3\dot{F}_{33z} & +\bar{A}_3\dot{F}_{43z} & +\bar{A}_3\dot{F}_{53z} & +\bar{A}_3\dot{F}_{63z} & N_3^{(3)}\sin\alpha_3 & +N_3\cos\alpha_3 & N_2^{(3)}\sin\alpha_3) & +N_3\cos\alpha_3) & & \\
 & & & & & & -S_3^{(3)}\cos\alpha_3+ & -S_3^{(3)}\sin\alpha_3- & -S_3^{(3)}\cos\alpha_3+ & -S_3^{(3)}\sin\alpha_3- & & \\
 & & & & & & +N_3^{(3)}\sin\alpha_3 & N_3^{(3)}\cos\alpha_3 & +N_3^{(3)}\sin\alpha_3 & N_3^{(3)}\cos\alpha_3 & &
\end{array}\right.
\end{array}
\quad (12.3
$$

The H1 matrix is written as a triply subscripted variable.

$$
\begin{aligned}
\overline{H1}_{1jk} &= \dot{F}_{k12}\,\dot{F}_{j11} + \dot{F}_{k11}\,\dot{F}_{j12} \\
\overline{H1}_{2jk} &= \dot{F}_{k12}\,\dot{F}_{j12} \\
\overline{H1}_{3jk} &= \dot{F}_{k22}\,\dot{F}_{j21} + \dot{F}_{k21}\,\dot{F}_{j22} \\
\overline{H1}_{4jk} &= \dot{F}_{k22}\,\dot{F}_{j22} \\
\overline{H1}_{5jk} &= \dot{F}_{k32}\,\dot{F}_{j31} + \dot{F}_{k31}\,\dot{F}_{j32} \\
\overline{H1}_{6jk} &= \dot{F}_{k32}\,\dot{F}_{j32} \\
\overline{H1}_{7jk} &= \bar{F}_{k12}\,\bar{F}_{j11} + \bar{F}_{k11}\,\bar{F}_{j12} \\
\overline{H1}_{8jk} &= \bar{F}_{k12}\,\bar{F}_{j12} \\
\overline{H1}_{9jk} &= \bar{F}_{k22}\,\bar{F}_{j21} + \bar{F}_{k21}\,\bar{F}_{j22} \\
\overline{H1}_{10jk} &= \bar{F}_{k22}\,\bar{F}_{j22} \\
\overline{H1}_{11jk} &= \bar{F}_{k32}\,\bar{F}_{j31} + \bar{F}_{k31}\,\bar{F}_{j32} \\
\overline{H1}_{12jk} &= \bar{F}_{k32}\,\bar{F}_{j32} \\
\overline{H1}_{13jk} &= \dot{F}_{k12}\,\bar{F}_{j11} + \dot{F}_{k11}\,\bar{F}_{j12} + \bar{F}_{k12}\,\dot{F}_{j11} + \bar{F}_{k11}\,\dot{F}_{j12} \\
\overline{H1}_{14jk} &= \dot{F}_{k12}\,\bar{F}_{j12} + \bar{F}_{k12}\,\dot{F}_{j12} \\
\overline{H1}_{15jk} &= \dot{F}_{k22}\,\bar{F}_{j21} + \dot{F}_{k21}\,\bar{F}_{j22} + \bar{F}_{k22}\,\dot{F}_{j21} + \bar{F}_{k21}\,\dot{F}_{j22} \\
\overline{H1}_{16jk} &= \dot{F}_{k22}\,\bar{F}_{j22} + \bar{F}_{k22}\,\dot{F}_{j22} \\
\overline{H1}_{17jk} &= \dot{F}_{k32}\,\bar{F}_{j31} + \dot{F}_{k31}\,\bar{F}_{j32} + \bar{F}_{k32}\,\dot{F}_{j31} + \bar{F}_{k31}\,\dot{F}_{j32} \\
\overline{H1}_{18jk} &= \dot{F}_{k32}\,\bar{F}_{j32} + \bar{F}_{k32}\,\dot{F}_{j32}
\end{aligned}
\tag{12.37}
$$

The variables used to define these matrices are:

$$
\begin{aligned}
\dot{A}_1 &= A_1\left(N_3^{(1)} - N_2^{(1)}\right) + A_2\left(-N_1^{(1)}\right) + A_3 N_1^{(1)} \\
\bar{A}_1 &= A_1\left(S_3^{(1)} - S_2^{(1)}\right) + A_2\left(-S_1^{(1)}\right) + A_3 S_1^{(1)} \\
\dot{A}_2 &= A_1 N_2^{(2)} + A_2\left(N_1^{(2)} - N_3^{(2)}\right) + A_3\left(-N_2^{(2)}\right) \\
\bar{A}_2 &= A_1 S_2^{(2)} + A_2\left(S_1^{(2)} - S_3^{(2)}\right) + A_3\left(-S_2^{(2)}\right) \\
\dot{A}_3 &= A_1\left(-N_3^{(3)}\right) + A_2 N_3^{(3)} + A_3\left(N_2^{(3)} - N_1^{(3)}\right) \\
\bar{A}_4 &= A_1\left(-S_3^{(3)}\right) + A_2 S_3^{(3)} + A_3\left(S_2^{(3)} - S_1^{(3)}\right)
\end{aligned}
\qquad (12.38)
$$

See Table 2 for an illustration of the following area coordinate transformations.

$$
\begin{aligned}
N_1^{(1)} &= 0 & S_1^{(1)} &= s_3^{(1)} \\
N_2^{(1)} &= -n_1^{(1)} & S_2^{(1)} &= s_1^{(1)} - s_3^{(1)} \\
N_3^{(1)} &= n_1^{(1)} & S_3^{(1)} &= -s_1^{(1)} \\
\\
N_1^{(2)} &= n_2^{(2)} & S_1^{(2)} &= -s_2^{(2)} \\
N_2^{(2)} &= 0 & S_2^{(2)} &= s_1^{(2)} \\
N_3^{(2)} &= -n_2^{(2)} & S_3^{(3)} &= s_2^{(2)} - s_1^{(2)} \\
\\
N_1^{(3)} &= -n_3^{(3)} & S_1^{(3)} &= s_3^{(3)} - s_2^{(3)} \\
N_2^{(3)} &= n_3^{(3)} & S_2^{(3)} &= -s_3^{(3)} \\
N_3^{(3)} &= 0 & S_3^{(3)} &= s_2^{(3)}
\end{aligned}
\qquad (12.39)
$$

$$\bar{F}_{111} = \frac{1}{2} J_1 S_1^{(1)} \tag{12.40}$$

$$\bar{F}_{331} = \frac{1}{2} J_3 S_3^{(3)}$$

$$\bar{F}_{521} = \frac{1}{2} J_5 S_2^{(2)}$$

$$\bar{F}_{112} = -\frac{1}{2} J_1 S_1^{(1)}$$

$$\bar{F}_{332} = -\frac{1}{2} J_3 S_3^{(3)}$$

$$\bar{F}_{522} = -\frac{1}{2} J_5 S_2^{(2)}$$

$$\bar{F}_{131} = \frac{1}{2} J_1 S_3^{(3)}$$

$$\bar{F}_{321} = 2 y_3 S_3^{(2)} + \frac{1}{2} J_3 S_2^{(2)}$$

$$\bar{F}_{511} = -2 y_3 S_2^{(1)} + \frac{1}{2} J_5 S_1^{(1)}$$

$$\bar{F}_{132} = -y_3 S_3^{(3)} - \frac{1}{2} J_1 S_3^{(3)}$$

$$\bar{F}_{322} = -2 y_3 S_3^{(2)} - \frac{1}{2} J_3 S_2^{(2)} + y_3 S_1^{(2)} + y_3 S_2^{(2)}$$

$$\bar{F}_{512} = 2 y_3 S_2^{(1)} - \frac{1}{2} J_5 S_1^{(1)} - y_3 S_3^{(1)}$$

$$\bar{F}_{120} = -y_3 S_3^{(2)}$$

$$\bar{F}_{310} = y_3 S_1^{(1)} + y_3 S_2^{(1)}$$

$$\bar{F}_{530} = -y_3 S_3^{(3)}$$

$$\bar{F}_{121} = 2 y_3 S_3^{(2)} - 2 y_3 S_1^{(2)} + \frac{1}{2} J_1 S_2^{(2)}$$

$$\bar{F}_{311} = 2 y_3 S_3^{(1)} + \frac{1}{2} J_3 S_1^{(1)} - 2 y_3 S_1^{(1)} - 2 y_3 S_2^{(1)}$$

$$\bar{F}_{531} = 2 y_3 S_3^{(3)} + \frac{1}{2} J_5 S_3^{(3)}$$

$$\bar{F}_{122} = 2 y_3 S_1^{(2)} - \frac{1}{2} J_1 S_2^{(2)} - y_3 S_3^{(2)}$$

$$\bar{F}_{312} = y_3 S_1^{(1)} + y_3 S_2^{(1)} - 2 y_3 S_3^{(1)} - \frac{1}{2} J_3 S_1^{(1)}$$

$$\bar{F}_{532} = -y_3 S_3^{(3)} - \frac{1}{2} J_5 S_3^{(3)}$$

$$\bar{F}_{211} = \frac{1}{2} J_2 S_1^{(1)}$$

$$\bar{F}_{431} = \frac{1}{2} J_4 S_3^{(3)}$$

$$\bar{F}_{621} = \frac{1}{2} J_6 S_2^{(2)}$$

$$
\begin{aligned}
\bar{F}_{212} &= -\tfrac{1}{2} J_2 S_1^{(1)} \\
\bar{F}_{432} &= -\tfrac{1}{2} J_4 S_3^{(3)} \\
\bar{F}_{622} &= -\tfrac{1}{2} J_6 S_2^{(2)} \\
\bar{F}_{220} &= X_2 S_2^{(2)} + X_3 S_3^{(2)} \\
\bar{F}_{410} &= -X_3 S_1^{(1)} + X_{23} S_2^{(1)} \\
\bar{F}_{630} &= X_{32} S_3^{(3)} - X_2 S_1^{(3)} \\
\bar{F}_{221} &= 2X_3 S_1^{(2)} + \tfrac{1}{2} J_2 S_2^{(2)} - 2X_2 S_2^{(2)} - 2X_3 S_3^{(2)} \\
\bar{F}_{411} &= 2X_{23} S_3^{(1)} + \tfrac{1}{2} J_4 S_1^{(1)} + 2X_3 S_1^{(1)} - 2X_{23} S_2^{(1)} \\
\bar{F}_{631} &= -2X_{32} S_3^{(3)} + 2X_2 S_1^{(3)} - 2X_2 S_2^{(3)} + \tfrac{1}{2} J_6 S_3^{(3)} \\
\bar{F}_{222} &= -2X_3 S_1^{(2)} - \tfrac{1}{2} J_2 S_2^{(2)} + X_2 S_2^{(2)} + X_3 S_3^{(2)} \\
\bar{F}_{412} &= -2X_{23} S_3^{(1)} - \tfrac{1}{2} J_4 S_1^{(1)} - X_3 S_1^{(1)} + X_{23} S_2^{(1)} \\
\bar{F}_{632} &= 2X_2 S_2^{(3)} - \tfrac{1}{2} J_6 S_3^{(3)} + X_{32} S_3^{(3)} - X_2 S_1^{(3)} \\
\bar{F}_{231} &= 2X_2 S_1^{(3)} + \tfrac{1}{2} J_2 S_3^{(3)} \\
\bar{F}_{421} &= -2X_3 S_3^{(2)} + \tfrac{1}{2} J_4 S_2^{(2)} \\
\bar{F}_{611} &= 2X_{32} S_2^{(1)} + \tfrac{1}{2} J_6 S_1^{(1)} \\
\bar{F}_{232} &= X_2 S_2^{(3)} + X_3 S_3^{(3)} - 2X_2 S_1^{(3)} - \tfrac{1}{2} J_2 S_3^{(3)} \\
\bar{F}_{422} &= 2X_3 S_3^{(2)} - \tfrac{1}{2} J_4 S_2^{(2)} - X_3 S_1^{(2)} + X_{23} S_2^{(2)}, \\
\bar{F}_{612} &= -2X_{32} S_2^{(1)} - \tfrac{1}{2} J_6 S_1^{(1)} + X_{32} S_3^{(1)} - X_2 S_1^{(1)}
\end{aligned}
\qquad (12.40)\ \text{(CONT.)}
$$

$$
\left.\begin{aligned}
J_i &= y_{i,i+1} + y_{i,i+2} = 2y_i - y_{i+1} - y_{i+2} \\
J_{i+1} &= X_{i+1} + X_{i+2} - 2X_i
\end{aligned}\right\} \quad i = 1, 3, 5 \qquad (12.41)
$$

The parameters $\dot{F}_{im1}$ and $\dot{F}_{im2}$ are defined to be the same as F_{im1} and F_{im2}, except replace $S_i^{(m)}$ by $N_i^{(m)}$.

$$
\begin{Bmatrix} \omega_1 \\ \theta^*_{\omega x_1} \\ \theta^*_{\omega y_1} \\ \omega_2 \\ \theta^*_{\omega x_2} \\ \theta_{\omega y_2} \\ \omega_3 \\ \theta_{\omega x_3} \\ \theta_{\omega y_3} \end{Bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{x_{32}}{2A} & -1 & 0 & \frac{x_{13}}{2A} & 0 & 0 & \frac{x_{21}}{2A} & 0 & 0 \\
-\frac{y_{23}}{2A} & 0 & -1 & -\frac{y_{31}}{2A} & 0 & 0 & -\frac{y_{12}}{2A} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{x_{32}}{2A} & 0 & 0 & \frac{x_{13}}{2A} & -1 & 0 & \frac{x_{21}}{2A} & 0 & 0 \\
-\frac{y_{23}}{2A} & 0 & 0 & -\frac{y_{31}}{2A} & 0 & -1 & -\frac{y_{12}}{2A} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{x_{32}}{2A} & 0 & 0 & \frac{x_{13}}{2A} & 0 & 0 & \frac{x_{21}}{2A} & -1 & 0 \\
-\frac{y_{23}}{2A} & 0 & 0 & -\frac{y_{31}}{2A} & 0 & 0 & -\frac{y_{12}}{2A} & 0 & -1
\end{bmatrix}
\begin{Bmatrix} \omega_1 \\ \theta_{\omega x_1} \\ \theta_{\omega y_1} \\ \omega_2 \\ \theta_{\omega x_2} \\ \theta_{\omega y_2} \\ \omega_3 \\ \theta_{\omega x_3} \\ \theta_{\omega y_3} \end{Bmatrix}
\quad (12.42)
$$

or

$$
\begin{Bmatrix} \omega_1 \\ \theta_{\omega x_1} \\ \vdots \\ \theta_{\omega y_3} \end{Bmatrix} = [T_\omega] \begin{Bmatrix} \omega_1 \\ \theta_{\omega x_1} \\ \vdots \\ \theta_{\omega y_3} \end{Bmatrix} \quad (12.43)
$$

Similarly for

$$
\begin{Bmatrix} u_1 \\ \theta^*_{u x_1} \\ \vdots \\ \theta^*_{u y_3} \end{Bmatrix} = [T_u] \begin{Bmatrix} u_1 \\ \theta_{u x_1} \\ \vdots \\ \theta_{u y_3} \end{Bmatrix} \quad (12.44)
$$

where $[T_u]$ is the same as $[T_w]$ except $T_{u_{22}} = T_{u_{33}} = T_{u_{55}} = T_{u_{66}} = T_{u_{88}} = T_{u_{99}} = 1$ instead of -1

$$\begin{Bmatrix} V_1 \\ \theta^*_{vx1} \\ \vdots \\ \theta^*_{vy3} \end{Bmatrix} = [T_v] \begin{Bmatrix} V_1 \\ \theta_{vx1} \\ \vdots \\ \theta_{vy3} \end{Bmatrix} \tag{12.45}$$

where $[T_v]$ is the same as $[T_w]$ except $T_{v33} = T_{v66} = T_{v99} = 1$ instead of -1.

These matrices $[T_u]$, $[T_v]$ and $[T_w]$ must be reordered to form

$$\{\delta^*\} = [T^*]\{q\} \tag{12.46}$$

12.7 QUADRILATERAL ELEMENT GO MATRIX

$$GO_{i,j} = GO_{i,j+1} = GO_{i,j+2} = GO_{i,j+3} = GO_{i,j+4} \qquad (12.47)$$

$i = 1, 3, 5, 9, 11$:

$$GO_{i,j} = \frac{1}{4}\xi_n(1+\eta_n\eta)(2\xi_n\xi+\eta_n\eta) \qquad \begin{Bmatrix} j = 1, 11, 21, 31 \\ n = 1, 3, 5, 7 \end{Bmatrix}$$

$$= -\xi(1+\eta_n\eta) \qquad \begin{Bmatrix} j = 6, 26 \\ n = 2, 6 \end{Bmatrix}$$

$$= \frac{1}{2}\xi_n(1-\eta^2) \qquad \begin{Bmatrix} j = 16, 36 \\ n = 4, 8 \end{Bmatrix}$$

$i = 2, 4, 6, 10, 12$:

$$GO_{i,j} = \frac{1}{4}\eta_n(1+\xi_n\xi)(\xi_n\xi+2\eta_n\eta) \qquad \begin{Bmatrix} j = 1, 11, 21, 31 \\ n = 1, 3, 5, 7 \end{Bmatrix}$$

$$= \frac{1}{2}\eta_n(1-\xi^2) \qquad \begin{Bmatrix} j = 6, 26 \\ n = 2, 6 \end{Bmatrix}$$

$$= -\eta(1+\xi_n\xi) \qquad \begin{Bmatrix} j = 16, 36 \\ n = 4, 8 \end{Bmatrix}$$

NOTE: j indicates column index in $[GO_{ij}]$,

n indicates value of coordinate at node number n.

$j = 1, 11, 21 \ldots$
$n = 1, 3, 5 \ldots$ etc. means use $n = 1$ if $j = 1$,

" $n = 3$ if $j = 11$,

" $n = 5$ if $j = 21$,

etc.

[GO] Matrix - AZI Function Contributions, cont'd.

(12.47)
(CONT.)

$$GO_{i,j} = GO_{i,j+1}$$

$i = 7, 8$:

$$GO_{i,j} = \tfrac{1}{4}(1+\xi_n\xi)(1+\eta_n\eta)(\xi_n\xi+\eta_n\eta-1) \qquad \left\{\begin{matrix} j = 4, 14, 24, 34 \\ n = 1, 3, 5, 7 \end{matrix}\right\}$$

$$= \tfrac{1}{2}(1-\xi^2)(1+\eta_n\eta) \qquad \left\{\begin{matrix} j = 9, 29 \\ n = 2, 6 \end{matrix}\right\}$$

$$= \tfrac{1}{2}(1-\eta^2)(1+\xi_n\xi) \qquad \left\{\begin{matrix} j = 19, 39 \\ n = 4, 8 \end{matrix}\right\}$$

[GO] Matrix - HMN Function Contributions

Let $G = G(\xi, \eta)$ represent u

$\bar{G} = G(\eta, \xi)$ represent v

$$G = \sum_{n=1}^{9} a_n G_n(\xi, \eta), \quad \bar{G} = \sum_{n=1}^{9} b_n G_n(\eta, \xi) = \sum_{n=1}^{9} b_n \bar{G}_n(\xi, \eta) \tag{12.48}$$

$$\frac{\partial G}{\partial \xi} = \sum_{n=1}^{9} a_n \frac{\partial G_n}{\partial \xi}, \quad \frac{\partial \bar{G}}{\partial \xi} = \sum_{n=1}^{9} b_n \frac{\partial \bar{G}_n}{\partial \xi}, \tag{12.49}$$

n	G_n	$\bar{G}_n$	
1	$(\xi - \xi^3)(1-\eta)$	$(\eta - \eta^3)(1-\xi)$	
2	$(\xi - \xi^3)(1+\eta)$	$(\eta - \eta^3)(1+\xi)$	
3	$2(\xi - \xi)(1-\eta^2)$	$2(\eta - \eta^3)(1-\xi^2)$	
4	$(1-\xi^2)(1-\eta^2)$	$(1-\eta^2)(1-\xi^2)$	(12.50)
5	$2(1-\xi^2)(\eta - \eta^3)$	$2(1-\eta^2)(\xi - \xi^3)$	
6	$\frac{1}{2}(\xi-1)(1-\eta^2)(\eta^2)$	$\frac{1}{2}(\eta-1)(1-\xi^2)(\xi^2)$	
7	$\frac{1}{2}(\xi+1)(1-\eta^2)(\eta^2)$	$\frac{1}{2}(\eta+1)(1-\xi^2)(\xi^2)$	
8	$\frac{1}{4}(\xi-1)(1-\eta^2)(\eta)$	$\frac{1}{4}(\eta-1)(1-\xi^2)(\xi)$	
9	$\frac{1}{4}(\xi+1)(1-\eta^2)(\eta)$	$\frac{1}{4}(\eta+1)(1-\xi^2)(\xi)$	

[GO] Matrix - HMN Function Contributions, cont'd.

n,m	$\frac{\partial G_n}{\partial \xi}$	$\frac{\partial \bar{G}_m}{\partial \eta}$
1	$(1-3\xi^2)(1-\eta)$	$(1-3\eta^2)(1-\xi)$
2	$(1-3\xi^2)(1+\eta)$	$(1-3\eta^2)(1+\xi)$
3	$2(1-3\xi^2)(1-\eta^2)$	$2(1-3\eta^2)(1-\xi^2)$
4	$-2\xi(1-\eta^2)$	$-2\eta(1-\xi^2)$
5	$-4\xi(\eta-\eta^3)$	$-4\eta(\xi-\xi^3)$
6	$\frac{1}{2}(\eta^2-\eta^4)$	$\frac{1}{2}(\xi^2-\xi^4)$
7	$\frac{1}{2}(\eta^2-\eta^4)$	$\frac{1}{2}(\xi^2-\xi^4)$
8	$\frac{1}{4}(\eta-\eta^3)$	$\frac{1}{4}(\xi-\xi^3)$
9	$\frac{1}{4}(\eta-\eta^3)$	$\frac{1}{4}(\xi-\xi^3)$

(12.51)

n,m	$\frac{\partial G_n}{\partial \eta}$	$\frac{\partial \bar{G}_m}{\partial \xi}$
1	$-(\xi-\xi^3)$	$-(\eta-\eta^3)$
2	$(\xi-\xi^3)$	$(\eta-\eta^3)$
3	$-4\eta(\xi-\xi^3)$	$-4\xi(\eta-\eta^3)$
4	$-2\eta(1-\xi^2)$	$-2\xi(1-\eta^2)$
5	$2(1-3\eta^2)(1-\xi^2)$	$2(1-3\xi^2)(1-\eta^2)$
6	$(\eta-2\eta^3)(\xi-1)$	$(\xi-2\xi^3)(\eta-1)$
7	$(\eta-2\eta^3)(\xi+1)$	$(\xi-2\xi^3)(\eta+1)$
8	$\frac{1}{4}(1-3\eta^2)(\xi-1)$	$\frac{1}{4}(1-3\xi^2)(\eta-1)$
9	$\frac{1}{4}(1-3\eta^2)(\xi+1)$	$\frac{1}{4}(1-3\xi^2)(\eta+1)$

(12.52)

[GO] Matrix - HMN Function Contributions, cont'd.

Columns 41 thru 8:

j, m, n correspond according to following table.

j =	41	42	43	44	45	46	47	48	49
n =	4	5	8	9	-	-	-	-	-
m =	-	-	-	-	4	5	8	9	-
j =	50	51	52	53	54	55	56	57	58
n =	2	3	-	-	7	6	-	-	-
m =	-	-	6	7	-	-	1	2	3

$$GO_{1j} = \frac{\partial G_n}{\partial \xi}$$

$$GO_{2j} = \frac{\partial G_n}{\partial \eta}$$

$$GO_{3j} = \frac{\partial \bar{G}_m}{\partial \xi} \qquad (12.53)$$

$$GO_{4j} = \frac{\partial \bar{G}_m}{\partial \eta}$$

12.8 THE TRANSFORMATION $\bar{J}$

The transformation required is

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} \qquad (12.54)$$

where the matrix $[J]$ is known from Equation

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \qquad (12.55)$$

Therefore evaluate $[J]$ at each of the integration points, and invert to obtain the elements of $[\bar{J}]$ at each of the integration points.

$$[J]^{-1} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} = \begin{bmatrix} \xi_x & \eta_x & \cancelto{0}{\zeta_x} \\ \xi_y & \cancelto{0}{\eta_y} & \cancelto{0}{\zeta_y} \\ \cancelto{0}{\xi_z} & \cancelto{0}{\eta_z} & \cancelto{1}{\zeta_z} \end{bmatrix} \qquad (12.56)$$

$$\{\theta\} = [\,\bar{J}\,]\{\bar{\theta}\}$$

$$
\begin{Bmatrix}
\partial u/\partial x \\
\partial u/\partial y \\
\partial v/\partial x \\
\partial v/\partial y \\
\partial w/\partial x \\
\partial w/\partial y \\
\theta_x \\
\theta_y \\
\partial \theta_x/\partial x \\
\partial \theta_x/\partial y \\
\partial \theta_y/\partial x \\
\partial \theta_y/\partial y
\end{Bmatrix}
=
\begin{bmatrix}
j_{11} & j_{12} & & & & & & & & & & \\
j_{21} & j_{22} & & & & & & & & & & \\
& & j_{11} & j_{12} & & & & & & & & \\
& & j_{21} & j_{22} & & & & & & & & \\
& & & & j_{11} & j_{12} & & & & & & \\
& & & & j_{21} & j_{22} & & & & & & \\
& & & & & & 1 & & & & & \\
& & & & & & & 1 & & & & \\
& & & & & & & & j_{11} & j_{12} & & \\
& & & & & & & & j_{21} & j_{22} & & \\
& & & & & & & & & & j_{11} & j_{12} \\
& & & & & & & & & & j_{21} & j_{22}
\end{bmatrix}
\begin{Bmatrix}
\partial u/\partial \xi \\
\partial u/\partial \eta \\
\partial v/\partial \xi \\
\partial v/\partial \eta \\
\partial w/\partial \xi \\
\partial w/\partial \eta \\
\theta_x \\
\theta_y \\
\partial \theta_\xi/\partial \xi \\
\partial \theta_\xi/\partial \eta \\
\partial \theta_\eta/\partial \xi \\
\partial \theta_\eta/\partial \eta
\end{Bmatrix}
\qquad (12.57)
$$

12.9 QUADRILATERAL ELEMENT A0 & A1 MATRICES

$$\lfloor \theta \rfloor = \lfloor u_x, u_y, v_x, v_y, \omega_x, \omega_y, \theta_x, \theta_y, \theta_{xx}, \theta_{xy}, \theta_{yx}, \theta_{yy} \rfloor \quad (12.59)$$

$$\lfloor \Delta \varepsilon \rfloor = \lfloor \Delta E_{xx}, \Delta E_{yy}, \Delta E_{xy}, \Delta E_{xz}, \Delta E_{yz}, \frac{\Delta E_{xx}^{(B)}}{z}, \frac{\Delta E_{yy}^{(B)}}{z}, \frac{\Delta E_{xy}^{(B)}}{z} \rfloor \quad (12.60)$$

$$\Delta E_i = (A0_{ij} + A1_{ijk}\theta_k)\Delta\theta_j \quad (12.61)$$

$$A0_{1,1} = 1 \qquad A1_{1,5,5} = 1$$
$$A0_{1,5} = \frac{\partial \omega^0}{\partial x}$$

$$A0_{2,4} = 1 \qquad A1_{2,6,6} = 1$$
$$A0_{2,6} = \frac{\partial \omega^0}{\partial y}$$

$$(12.62)$$

$$A0_{3,2} = 1 \qquad A1_{2,5,6} = 1$$
$$A0_{3,3} = 1 \qquad A1_{3,6,5} = 1$$
$$A0_{3,5} = + \frac{\partial \omega^0}{\partial y}$$
$$A0_{3,6} = + \frac{\partial \omega^0}{\partial x}$$

$$A0_{4,5} = 1 \qquad A0_{4,1} = -\frac{\partial \omega^0}{\partial x} \qquad A1_{4,8,1} = 1 \qquad A1_{4,1,8} = 1$$
$$A0_{4,8} = 1 \qquad A0_{4,3} = -\frac{\partial \omega^0}{\partial y} \qquad A1_{4,7,3} = -1 \qquad A1_{4,3,7} = -1$$
$$A0_{5,6} = 1 \qquad A0_{5,2} = -\frac{\partial \omega^0}{\partial x} \qquad A1_{5,8,2} = 1 \qquad A1_{5,2,8} = 1$$
$$A0_{5,7} = -1 \qquad A0_{5,4} = -\frac{\partial \omega^0}{\partial y} \qquad A1_{5,7,4} = -1 \qquad A1_{5,4,7} = -1$$

$$\Delta E_i^{(6)} = \Sigma \left(A0_{ij} + A1_{ijk}\,\theta_i \right) \Delta\theta_j \qquad i \geq 5 \tag{12.63}$$

$$\begin{array}{lll}
A0_{6,1} = -\dfrac{\partial^2 w^0}{\partial x^2} & A1_{6,11,1} = 1 & A1_{6,1,11} = 1 \\
A0_{6,3} = -\dfrac{\partial^2 w^0}{\partial x\,\partial y} & A1_{6,9,3} = -1.0 & A1_{6,3,9} = -1.0 \\
A0_{6,11} = 1 & & \\
 & & \\
A0_{7,2} = -\dfrac{\partial^2 w^0}{\partial x\,\partial y} & A1_{7,10,4} = -1 & A1_{7,4,10} = -1 \\
A0_{7,4} = -\dfrac{\partial^2 w^0}{\partial y^2} & & \\
A0_{7,10} = -1 & A1_{7,12,2} = 1.0 & A1_{7,2,12} = 1.0 \\
 & & \\
A0_{8,1} = -\dfrac{\partial^2 w^0}{\partial x\,\partial y} & & \\
A0_{8,2} = -\dfrac{\partial^2 w^0}{\partial x^2} & A1_{8,12,1} = 1 & A1_{8,1,12} = 1 \\
A0_{8,3} = -\dfrac{\partial^2 w^0}{\partial y^2} & A1_{8,9,4} = -1 & A1_{8,4,9} = -1 \\
A0_{8,4} = -\dfrac{\partial^2 w^0}{\partial x\,\partial y} & & \\
A0_{8,9} = -1 & A1_{8,11,2} = 1 & A1_{8,2,11} = 1 \\
A0_{8,12} = -1 & A1_{8,10,3} = -1 & A1_{8,3,10} = -1
\end{array} \tag{12.64}$$

12.10 THE QUADRILATERAL ELEMENT [DO] MATRIX

$$
[DO] = \begin{bmatrix}
\frac{Et}{1-\nu^2} & \frac{E\nu t}{1-\nu^2} & & & & & & \\
\frac{E\nu t}{1-\nu^2} & \frac{Et}{1-\nu^2} & & & & & & \\
 & & \frac{Et}{2(1+\nu)} & & & & & \\
 & & & \frac{Et}{2(1+\nu)} & & & & \\
 & & & & \frac{Et}{2(1+\nu)} & & & \\
 & & & & & \frac{Et^3}{12(1-\nu^2)} & \frac{E\nu t^3}{12(1-\nu^2)} & \\
 & & & & & \frac{E\nu t^3}{12(1-\nu^2)} & \frac{Et^3}{12(1-\nu^2)} & \\
 & & & & & & & \frac{Et^3}{24(1+\nu)}
\end{bmatrix} \qquad (12.65)
$$

12.11 QUADRILATERAL ELEMENT HMN EQUATIONS

HMN Equations LHS

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} a_1 & a_2 & a_3 & b_6 & b_7 & a_7 & a_6 & b_1 & b_2 & b_3 \end{array} \\
[H] = & \left[\begin{array}{cccccccccc}
0 & 1 & 0 & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & 1 & & & & & & & \\
1 & 0 & 0 & & & & & & & \\
-\frac{1}{4} & \frac{1}{4} & -1 & 0 & 1 & & & & & \\
-\frac{1}{4} & \frac{1}{4} & 1 & -1 & 0 & & & & & \\
 & & & & & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & -1 \\
 & & & & & 0 & -1 & -\frac{1}{4} & \frac{1}{4} & 1 \\
 & & & & & & & 0 & 1 & 0 \\
 & & & & & & & \frac{1}{2} & \frac{1}{2} & 1 \\
 & & & & & & & 1 & 0 & 0
\end{array}\right]
\end{array}
\qquad (12.66)
$$

Then defining

(ξ_m, η_m) the values of ξ and η at node m

$$
HO_{1M} = \begin{cases} \sum\limits_{1,3,5,7}^{n} \frac{1}{24} \omega_n^o \xi_n^2 \xi_M^2 (1+\eta_n)(1+\eta_M) - \sum\limits_{2,6}^{n} \frac{1}{12} \omega_n^o \xi_M^2 (1+\eta_n)(1+\eta_M) & M = 1,3,5,7 \\ \sum\limits_{2,6}^{n} \frac{1}{6} \omega_n^o (1+\eta_n)(1+\eta_M) - \sum\limits_{1,3,5,7}^{n} \frac{1}{12} \omega_n^o \xi_n^2 (1+\eta_n)(1+\eta_M) & M = 2,6 \end{cases}
$$

$$
HO_{2M} = \begin{cases} \sum\limits_{1,3,5,7}^{n} \frac{1}{24} \omega_n^o \xi_n^2 \xi_M^2 - \sum\limits_{2,6}^{n} \frac{1}{12} \omega_n^o \xi_M^2 & M = 1,3,5,7 \\ \sum\limits_{2,6}^{n} \frac{1}{6} \omega_n^o - \sum\limits_{1,3,5,7}^{n} \frac{1}{12} \omega_n^o \xi_n^2 & M = 2,6, \end{cases}
$$

$$
HO_{3M} = \begin{cases} \sum\limits_{1,3,5,7}^{n} \frac{1}{24} \omega_n^o \xi_n^2 \xi_M^2 (1-\eta_n)(1-\eta_M) - \sum\limits_{2,6}^{n} \frac{1}{12} \omega_n^o \xi_M^2 (1-\eta_n)(1-\eta_M) & M = 1,3,5,7 \\ \sum\limits_{2,6}^{n} \frac{1}{6} \omega_n^o (1-\eta_n)(1-\eta_M) - \sum\limits_{1,3,5,7}^{n} \frac{1}{12} \omega_n^o \xi_n^2 (1-\eta_n)(1-\eta_M) & M = 2,6 \end{cases} \tag{12.67}
$$

$$
HO_{4M} = \begin{cases} \sum\limits_{1,3,5,7}^{n} \frac{1}{32} \left[\xi_n^2 \xi_M \eta_M (1+\eta_n) + \xi_n^2 \xi_M^2 \eta_n (1+\eta_M) \right] \omega_n^o & \\ \qquad - \sum\limits_{2,6}^{n} \frac{1}{16} \left[\xi_M^2 \eta_n (1+\eta_M) + \xi_M^2 \eta_M (1+\eta_n) \right] \omega_n^o & M = 1,3,5,7 \\ \sum\limits_{2,6}^{n} \frac{1}{8} \left[\eta_M (1+\eta_n) + \eta_n (1+\eta_M) \right] \omega_n^o & \\ \qquad - \sum\limits_{1,3,5,7}^{n} \frac{1}{16} \left[\xi_n^2 \eta_M (1+\eta_n) + \xi_n^2 \eta_n (1+\eta_M) \right] \omega_n^o & M = 2,6 \end{cases}
$$

$$
HO_{5M} = \begin{cases} \sum\limits_{1,3,5,7}^{n} \frac{1}{32} \left[\xi_n^2 \xi_M^2 \eta_M (1-\eta_n) + \xi_n^2 \xi_M^2 \eta_n (1-\eta_M) \right] \omega_n^o & \\ \qquad - \sum\limits_{2,6}^{n} \frac{1}{16} \left[\xi_M^2 \eta_n (1-\eta_M) + \xi_M^2 \eta_M (1-\eta_n) \right] \omega_n^o & M = 1,3,5,7 \\ \sum\limits_{2,6}^{n} \frac{1}{8} \left[\eta_M (1-\eta_n) + \eta_n (1-\eta_M) \right] \omega_n^o & \\ \qquad - \sum\limits_{1,3,5,7}^{n} \frac{1}{16} \left[\xi_n^2 \eta_M (1-\eta_n) + \xi_n^2 \eta_n (1-\eta_M) \right] \omega_n^o & M = 2,6 \end{cases}
$$

$$HO_{6M} = \begin{cases} \displaystyle\sum_{1,3,5,7}^{n} \frac{1}{32}\left[\xi_n \eta_n^2 \eta_M^2 (1+\xi_M) + \xi_M \eta_n^2 \eta_M^2 (1+\xi_n)\right]\omega_n^0 - \sum_{4,8}^{n} \frac{1}{16}\left[\xi_M \eta_M^2 (1+\xi_n) + \xi_n \eta_M^2 (1+\xi_M)\right]\omega_n^0 & M = 1,3,5,7 \\ \displaystyle\sum_{4,8}^{n} \frac{1}{8}\left[\xi_n (1+\xi_M) + \xi_M (1+\xi_n)\right]\omega_n^0 - \sum_{1,3,5,7}^{n} \frac{1}{16}\left[\xi_n \eta_n^2 (1+\xi_M) + \xi_M \eta_n^2 (1+\xi_n)\right]\omega_n^0 & M = 4,8 \end{cases}$$

$$HO_{7M} = \begin{cases} \displaystyle\sum_{1,3,5,7}^{n} \frac{1}{32}\left[\xi_n \eta_n^2 \eta_M^2 (1-\xi_M) + \xi_M \eta_n^2 \eta_M^2 (1-\xi_n)\right]\omega_n^0 - \sum_{4,8}^{n} \frac{1}{16}\left[\xi_M \eta_M^2 (1-\xi_n) + \xi_n \eta_M^2 (1-\xi_M)\right]\omega_n^0 & M = 1,3,5,7 \\ \displaystyle\sum_{4,8}^{n} \frac{1}{8}\left[\xi_n (1-\xi_M) + \xi_M (1-\xi_n)\right]\omega_n^0 - \sum_{1,3,5,7}^{n} \frac{1}{16}\left[\xi_n \eta_n^2 (1-\xi_M) + \xi_M \eta_n^2 (1-\xi_n)\right]\omega_n^0 & M = 4,8 \end{cases} \quad (12.67)\ \text{(CONT.)}$$

$$HO_{8M} \begin{cases} \displaystyle\sum_{1,3,5,7}^{n} \frac{1}{24}\omega_n^0 \eta_n^2 \eta_M^2 (1+\xi_n)(1+\xi_M) - \sum_{4,8}^{n} \frac{1}{12}\eta_M^2 (1+\xi_n)(1+\xi_M)\omega_n^0 & M = 1,3,5,7 \\ \displaystyle\sum_{4,8}^{n} \frac{1}{6}(1+\xi_n)(1+\xi_M)\omega_n^0 - \sum_{1,3,5,7}^{n} \frac{1}{12}\eta_n^2 (1+\xi_n)(1+\xi_M)\omega_n^0 & M = 4,8 \end{cases}$$

$$HO_{9M} \begin{cases} \displaystyle\sum_{1,3,5,7}^{n} \frac{1}{24}\eta_n^2 \eta_M^2 \omega_n^0 - \sum_{4,8}^{n} \frac{1}{12}\eta_M^2 \omega_n^0 & M = 1,3,5,7 \\ \displaystyle\sum_{4,8}^{n} \frac{1}{6}\omega_n^0 - \sum_{1,3,5,7}^{n} \frac{1}{12}\eta_n^2 \omega_n^0 & M = 4,8 \end{cases}$$

$$HO_{10,M} \begin{cases} \displaystyle\sum_{1,3,5,7}^{n} \frac{1}{24}\eta_n^2 \eta_M^2 (1-\xi_n)(1-\xi_M)\omega_n^0 - \sum_{4,8}^{n} \frac{1}{12}\eta_M^2 (1-\xi_n)(1-\xi_M)\omega_n^0 & M = 1,3,5,7 \\ \displaystyle\sum_{4,8}^{n} \frac{1}{6}(1-\xi_n)(1-\xi_M)\omega_n^0 - \sum_{1,3,5,7}^{n} \frac{1}{12}\eta_n^2 (1-\xi_n)(1-\xi_M)\omega_n^0 & M = 4,8 \end{cases}$$

HMN Equations RHS ($\overline{H1}$) Note: $\overline{H1}_1 \rightarrow \overline{H1}_3$, $\overline{H1}_4 \rightarrow \overline{H1}_5$, $\overline{H1}_6 \rightarrow \overline{H1}_7$, $\overline{H1}_8 \rightarrow \overline{H1}_{10}$ by changing sign of $(1 \pm x_i)$ or $(1 \pm y_i)$

$$\overline{H1}_{1nM} = \begin{cases} \frac{1}{24}\,\xi_n^2\,\xi_M^2\,(1+\eta_n)(1+\eta_M) & n, M = 1,3,5,7 \\ \frac{1}{6}\,(1+\eta_n)(1+\eta_M) & n, M = 2,6 \\ -\frac{1}{12}\,\xi_n^2\,(1+\eta_n)(1+\eta_M) & n = 1,3,5,7 \quad M = 2,6 \\ -\frac{1}{12}\,\xi_M^2\,(1+\eta_n)(1+\eta_M) & n = 2,6 \quad M = 1,3,5,7 \end{cases}$$

$$\overline{H1}_{2nM} = \begin{cases} \frac{1}{24}\,\xi_n^2\,\xi_M^2 & n, M = 1,3,5,7 \\ \frac{1}{6} & n, M = 2,6 \\ -\frac{1}{12}\,\xi_n^2 & n = 1,3,5,7 \quad M = 2,6 \\ -\frac{1}{12}\,\xi_M^2 & n = 2,6 \quad M = 1,3,5,7 \end{cases}$$

$$\overline{H1}_{3nM} = \begin{cases} \frac{1}{24}\,\xi_n^2\,\xi_M^2\,(1-\eta_n)(1-\eta_M) & n, M = 1,3,5,7 \\ \frac{1}{6}\,(1-\eta_n)(1-\eta_M) & n, M = 2,6 \\ -\frac{1}{12}\,\xi_n^2\,(1-\eta_n)(1-\eta_M) & n = 1,3,5,7 \quad M = 2,6 \\ -\frac{1}{12}\,\xi_M^2\,(1-\eta_n)(1-\eta_M) & n = 2,6 \quad M = 1,3,5,7 \end{cases} \qquad (12.68)$$

$$\overline{H1}_{4nM} = \begin{cases} \frac{1}{32}\left[\xi_n^2\,\xi_M^2\,\eta_M\,(1+\eta_n) + \xi_n^2\,\xi_M^2\,\eta_n\,(1+\eta_M)\right] & n, M = 1,3,5,7 \\ \frac{1}{8}\left[\eta_M\,(1+\eta_n) + \eta_n\,(1+\eta_M)\right] & n, M = 2,6 \\ -\frac{1}{16}\left[\xi_n^2\,\eta_M\,(1+\eta_n) + \xi_n^2\,\eta_n\,(1+\eta_M)\right] & n = 1,3,5,7 \quad M = 2,6 \\ -\frac{1}{16}\left[\xi_M^2\,\eta_n\,(1-\eta_M) + \xi_M^2\,\eta_M\,(1+\eta_n)\right] & n = 2,6 \quad M = 1,3,5,7 \end{cases}$$

$$\overline{H1}_{5nM} = \begin{cases} \frac{1}{32}\left[\xi_n^2 \xi_M^2 \eta_M (1-\eta_n) + \xi_n^2 \xi_M^2 \eta_n (1-\eta_M)\right] & n, M = 1,3,5,7 \\ \frac{1}{8}\left[\eta_M (1-\eta_n) + \eta_n (1-\eta_M)\right] & n, M = 4,8 \\ -\frac{1}{16}\left[\xi_n^2 \eta_M (1-\eta_n) + \xi_n^2 \eta_n (1-\eta_M)\right] & n = 1,3,5,7 \quad M = 4,8 \\ -\frac{1}{16}\left[\xi_M^2 \eta_n (1-\eta_M) + \xi_M^2 \eta_M (1-\eta_n)\right] & n = 4,8 \quad M = 1,3,5,7 \end{cases}$$

$$\overline{H1}_{6nM} = \begin{cases} \frac{1}{32}\left[\xi_n \eta_n^2 \eta_M^2 (1+\xi_M) + \xi_M \eta_n^2 \eta_M^2 (1+\xi_n)\right] & n, M = 1,3,5,7 \\ \frac{1}{8}\left[\xi_n (1-\xi_M) + \xi_M (1-\xi_n)\right] & n, M = 4,8 \\ -\frac{1}{16}\left[\xi_n \eta_n^2 (1-\xi_M) + \xi_M \eta_n^2 (1-\xi_n)\right] & n = 1,3,5,7 \quad M = 4,8 \\ -\frac{1}{16}\left[\xi_M \eta_M^2 (1-\xi_n) + \xi_n \eta_M^2 (1-\xi_M)\right] & n = 4,8 \quad M = 1,3,5,7 \end{cases}$$

(12.68)
(CONT.)

$$\overline{H1}_{7nM} = \begin{cases} \frac{1}{32}\left[\xi_n \eta_n^2 \eta_M^2 (1-\xi_M) + \xi_M \eta_n^2 \eta_M^2 (1-\xi_M) \right. & n, M = 1,3,5,7 \\ \frac{1}{8}\left[\xi_n (1-\xi_M) + \xi_M (1-\xi_n)\right] & n, M = 4,8 \\ -\frac{1}{16}\left[\xi_n \eta_n^2 (1-\xi_M) + \xi_M \eta_n^2 (1-\xi_n)\right] & n = 1,3,7,9 \quad M = 4,8 \\ -\frac{1}{16}\left[\xi_M \eta_M^2 (1-\xi_n) + \xi_n \eta_M^2 (1-\xi_M)\right] & n = 4,8 \quad M = 1,3,7,9 \end{cases}$$

$$\overline{H1}_{8nM} = \begin{cases} \frac{1}{24} \eta_n^2 \eta_M^2 (1+\xi_n)(1+\xi_M) & n, M = 1,3,5,7 \\ \frac{1}{6} (1+\xi_n)(1+\xi_M) & n, M = 4,8 \\ -\frac{1}{12} \eta_n^2 (1+\xi_n)(1+\xi_M) & n = 1,3,5,7 \quad M = 4,8 \\ -\frac{1}{12} \eta_M^2 (1+\xi_n)(1+\xi_M) & n = 4,8 \quad M = 1,3,5,7 \end{cases}$$

$$\overline{H1}_{9nM} = \begin{cases} \frac{1}{24} \eta_n^2 \eta_M^2 & n, M = 1,3,5,7 \\ \frac{1}{6} & n, M = 4,8 \\ -\frac{1}{12} \eta_n^2 & n = 1,3,5,7 \quad M = 4,8 \\ -\frac{1}{12} \eta_M^2 & n = 4,8 \quad M = 1,3,5,7 \end{cases}$$

$$\overline{H1}_{10nM} \begin{cases} \frac{1}{24}\,\eta_n^2\,\eta_M^2(1-\xi_n)(1-\xi_M) & n,M = 1,3,5,7 \\ \frac{1}{6}\,(1-\xi_n)(1-\xi_M) & n,M = 4,8 \\ -\frac{1}{12}\,\eta_n^2(1-\xi_n)(1-\xi_M) & n = 1,3,5,7 \quad M = 4,8 \\ -\frac{1}{12}\,\eta_M^2(1-\xi_n)(1-\xi_M) & n = 4,8 \quad M = 1,3,5,7 \end{cases} \qquad \text{(12.68) (CONT.)}$$